Sample Problems of Experimental Mathematics

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Here is the complete list of the ten sample problems of experimental mathematics, as given in [1], together with answers and references. The introductory material on the ten SIAM problems, which was also included in Section 1.7 of [1], is also included for convenience. The authors wish to acknowledge Greg Fee of CECM for his assistance in some of these solutions.

The Ten SIAM Problems

In the January 2002 issue of *SIAM News*, Nick Trefethen of Oxford University presented ten diverse problems used in teaching graduate numerical analysis students at Oxford University, the answer to each being a certain real number. Readers were challenged to compute ten digits of each answer, with a $100 prize to the best entrant. Trefethen wrote, "If anyone gets 50 digits in total, I will be impressed."

Success in solving these problems requires a broad knowledge of mathematics and numerical analysis, together with significant computational effort to obtain solutions and ensure correctness of the results. The strengths and limitations of *Maple*, *Mathematica*, MATLAB and other software tools are strikingly revealed in these exercises.

A total of 94 teams, representing 25 different nations, submitted results. Twenty of these teams received a full 100 points (10 correct digits for each problem). Since these results were much better than expected, an anonymous donor provided funds for a $100 award to each team. The present authors and Greg Fee entered, but failed to qualify for an award. The ten problems are:

1. What is $\lim_{\epsilon \to 0} \int_\epsilon^1 x^{-1} \cos(x^{-1} \log x)\, dx$?

2. A photon moving at speed 1 in the x-y plane starts at $t = 0$ at $(x, y) = (1/2, 1/10)$ heading due east. Around every integer lattice point (i, j) in the plane, a circular mirror of radius 1/3 has been erected. How far from the origin is the photon at $t = 10$?

3. The infinite matrix A with entries $a_{11} = 1$, $a_{12} = 1/2$, $a_{21} = 1/3$, $a_{13} = 1/4$, $a_{22} = 1/5$, $a_{31} = 1/6$, etc., is a bounded operator on ℓ^2. What is $||A||$?

4. What is the global minimum of the function $\exp(\sin(50x))+\sin(60e^y)+\sin(70\sin x)+\sin(\sin(80y))-\sin(10(x+y))+(x^2+y^2)/4$?

5. Let $f(z)=1/\Gamma(z)$, where $\Gamma(z)$ is the gamma function, and let $p(z)$ be the cubic polynomial that best approximates $f(z)$ on the unit disk in the supremum norm $||\cdot||_\infty$. What is $||f-p||_\infty$?

6. A flea starts at $(0,0)$ on the infinite 2-D integer lattice and executes a biased random walk: At each step it hops north or south with probability $1/4$, east with probability $1/4+\epsilon$, and west with probability $1/4-\epsilon$. The probability that the flea returns to $(0,0)$ sometime during its wanderings is $1/2$. What is ϵ?

7. Let A be the 20000×20000 matrix whose entries are zero everywhere except for the primes $2,3,5,7,\cdots,224737$ along the main diagonal and the number 1 in all the positions a_{ij} with $|i-j|=1,2,4,8,\cdots,16384$. What is the $(1,1)$ entry of A^{-1}.

8. A square plate $[-1,1]\times[-1,1]$ is at temperature $u=0$. At time $t=0$ the temperature is increased to $u=5$ along one of the four sides while being held at $u=0$ along the other three sides, and heat then flows into the plate according to $u_t=\Delta u$. When does the temperature reach $u=1$ at the center of the plate?

9. The integral $I(a)=\int_0^2[2+\sin(10\alpha)]x^\alpha\sin(\alpha/(2-x))\,dx$ depends on the parameter α. What is the value $\alpha\in[0,5]$ at which $I(\alpha)$ achieves its maximum?

10. A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

These problems and their solutions are described in detail in a forthcoming book [3]. Answers correct to 40 digits are available at the URL
`http://web.comlab.ox.ac.uk/oucl/work/nick.trefethen/hundred.html`.

The Ten Experimental Math Problems

Inspired by this set of problems, the present authors have assembled a similar set of problems, similar in style to the SIAM/Oxford 100 Digit Challenge, but emphasizing the flavor of experimental mathematics. As in the above problem set, a real constant is defined in each case. The objective here is to produce at least 100 correct digits digits, so that a total of 1000 points can be earned. Several of these can be solved by fairly direct application of numerical computation, but others require mathematical analysis and reduction before computation can be done in reasonable time. Each problem provides an "extra credit" question, for which an additional 100 points may be earned. The maximum total score is thus 2000 points.

In each case, these problems can be solved with techniques presented either in this volume or in the companion volume. References to the relevant sections or exercises are included below.

1. Compute the value of r for which the chaotic iteration $x_{n+1} = rx_n(1 - x_n)$, starting with some $x_0 \in (0, 1)$, exhibits a bifurcation between 4-way periodicity and 8-way periodicity.

   Extra credit: This constant is an algebraic number of degree not exceeding 20. Find the minimal polynomial with integer coefficients that it satisfies.

2. Evaluate

$$\sum_{(m,n,p)\neq 0} \frac{(-1)^{m+n+p}}{\sqrt{m^2 + n^2 + p^2}}, \tag{1}$$

   where convergence means the limit of sums over the integer lattice points enclosed in increasingly large cubes surrounding the origin.

   Extra credit: Identify this constant.

3. Evaluate the sum

$$\sum_{k=1}^{\infty} \left(1 - \frac{1}{2} + \cdots + (-1)^{k+1}\frac{1}{k}\right)^2 (k+1)^{-3}.$$

   Extra credit: Evaluate this constant as a multiterm expression involving well-known mathematical constants. This expression has seven terms, and involves $\pi, \log 2, \zeta(3)$, and $\mathrm{Li}_5(1/2)$, where $\mathrm{Li}_n(x) = \sum_{k>0} x^n/n^k$. Hint: The expression is "homogenous," in the sense that each term has the same total "degree." The degrees of π and $\log 2$ are each 1, the degree of $\zeta(3)$ is 3, the degree of $\mathrm{Li}_5(1/2)$ is 5, and the degree of α^n is n times the degree of α.

4. Evaluate

$$\prod_{k=1}^{\infty} \left[1 + \frac{1}{k(k+2)}\right]^{\log_2 k} = \prod_{k=1}^{\infty} k^{\left[\log_2\left(1+\frac{1}{k(k+2)}\right)\right]}.$$

   Extra credit: Evaluate this constant in terms of a less-well-known mathematical constant.

5. Given $a, b, \eta > 0$, define

$$R_\eta(a,b) \quad = \quad \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}.$$

Calculate $R_1(2,2)$.

Extra credit: Evaluate this constant as a two-term expression involving a well-known mathematical constant.

6. Calculate the expected distance between two random points on different sides of the unit square.

Hint: This can be expressed in terms of integrals as

$$\frac{2}{3}\int_0^1\int_0^1 \sqrt{x^2+y^2}\,dx\,dy + \frac{1}{3}\int_0^1\int_0^1 \sqrt{1+(y-u)^2}\,du\,dy.$$

Extra credit: Express this constant as a three-term expression involving algebraic constants and an evaluation of the natural logarithm with an algebraic argument.

7. Calculate the expected distance between two random points on different faces of the unit cube.

Hint: This can be expressed in terms of integrals as

$$\begin{aligned} & \frac{4}{5}\int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{x^2+y^2+(z-w)^2}\,dw\,dx\,dy\,dz \\ + & \frac{1}{5}\int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{1+(y-u)^2+(z-w)^2}\,du\,dw\,dy\,dz. \end{aligned}$$

Extra credit: Express this constant as a six-term expression involving algebraic constants and two evaluations of the natural logarithm with algebraic arguments.

8. Calculate

$$\int_0^\infty \cos(2x)\prod_{n=1}^\infty \cos\left(\frac{x}{n}\right)\,dx.$$

Extra credit: Express this constant as an analytic expression. Hint: It is *not* what it first appears to be. See Chapter 2, Exercise 29 of the second volume.

9. Calculate

$$\sum_{i>j>k>l>0} \frac{1}{i^3 j k^3 l}.$$

   Extra credit: Express this constant as a single-term expression involving a well-known mathematical constant.

10. Evaluate

$$W_1 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{1}{3 - \cos(x) - \cos(y) - \cos(z)}\, dx\, dy\, dz.$$

   Extra credit: Express this constant in terms of the Gamma function.

Answers to Experimental Math Problems

1. Value:
   3.54409035955192285361596598660480454058309984544457367545781253030584294285886301225625856642489179999626...

   Extra credit: It is a root of

$$\begin{aligned} 0 &= 4913 + 2108t^2 - 604t^3 - 977t^4 + 8t^5 + 44t^6 + 392t^7 - 193t^8 - 40t^9 \\ &\quad + 48t^{10} - 12t^{11} + t^{12}. \end{aligned}$$

   See Section 2.3 of the first volume for additional information.

2. Value:
   $-1.74756459463318219063621203554439740348516143662474175815282535076504062353276117989075836269460788993\ldots$

   Extra credit: This is Madelung's constant, which is derived from analysis of the electric charge of a sodium chloride crystal. In particular, it is the potential at the origin assuming electric charges, alternating positive and negative, located precisely at all other cubic lattice positions. Alternative formulas that admit reasonably rapid computation are known for this and also for more general sums of this type. See Sections 4.3.2 and 4.3.3 of the second volume.

3. Value:
   0.156166933381176915881035909687988193685776709840303872957529354497075037440295791455205653709358147578...

Extra credit:

$$4\,\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{1}{30}\log^5(2) - \frac{17}{32}\zeta(5) - \frac{11}{720}\pi^4\log(2) + \frac{7}{4}\zeta(3)\log^2(2) + \frac{1}{18}\pi^2\log^3(2) - \frac{1}{8}\pi^2\zeta(3).$$

See Section 2.5 of the first volume and Section 3.5 of the second volume.

4. Value:
   2.685452001065306445309714835481795693820382293994462953051152345
   557218859537152002801141174931847697 9...

   Extra credit: This is Khintchine's constant, often denoted by K_0. Formulas for computing K_0 are given in Chapter 6, Exercise 2 of the first volume.

5. Value:
   0.429203673205103380768678308360248557901415300312447089512527703
   846091796856895500685982587328941466009...

   Extra credit: $2-\pi/2$. A general formula for $R_\eta(a,b)$ is given in Chapter 1, Exercise 53–54 of the second volume.

6. Value:
   0.869009055274534463884970594345406624856719279631680569660350864
   584179822174693053113213554875435754113...

   Extra credit:

   $$\frac{2}{9} + \frac{1}{9}\sqrt{2} + \frac{5}{9}\log\left(1+\sqrt{2}\right).$$

   This problem and the next are due to James D. Klein. See Chapter 1, Exercise 57 of the second volume.

7. Value:
   0.926390055174046729218163586547779014444960190107335046732521921
   271418504594036683829313473075349968212...

   Extra credit:

   $$\frac{4}{75} + \frac{17}{75}\sqrt{2} - \frac{2}{25}\sqrt{3} - \frac{7}{75}\pi + \frac{7}{25}\log\left(1+\sqrt{2}\right) + \frac{7}{25}\log\left(7+4\sqrt{3}\right).$$

8. Value:
   0.392699081698724154807830422909937860524645434187231595926812285
   162093247139938546179016512747455366777 ...

   Extra credit: It is very close to, but not equal to, $\pi/8$. See Chapter 2, Exercise 29 of the second volume.

9. Value:
   0.005229569563530960100930652283899231589890420784634635522547448
   972148869544660150074975454324856104016 27 ...

   Extra credit: $2\pi^8/10!$. In general, it can be shown that a $2n$-fold sum of the above type, where the powers in the denominator alternate 3 and 1, evaluates as $2\pi^{4n}/(4n+2)!$. See Section 3.7 of the second volume.

10. Value:
    15.67249523473857324485605345234758558194525720933521820276979922
    876186341583928633420288254038410687 70 ...

    Extra credit:

    $$\frac{1}{96}\,(\sqrt{3}-1)\,\Gamma^2\left(\frac{1}{24}\right)\,\Gamma^2\left(\frac{11}{24}\right).$$

    See Chapter 2, Exercise 20 of the second volume.

References

[1] Jonathan M. Borwein and David H. Bailey. *Mathematics by Experiment: Plausible Reasoning in the 21st Century*. A.K. Peters Ltd, Natick, MA, 2003.

[2] Jonathan M. Borwein and David H. Bailey. *Experimentation in Mathematics: Computational Paths to Discovery*. A.K. Peters Ltd, Natick, MA, 2003.

[3] Folkmar Bornemann, Dirk Laurie, Jörg Waldvogel, and Stan Wagon. *The SIAM/Oxford 100-digit Challenge*. Society for Industrial and Applied Mathematics, 2003 (to appear).