VIII. Experimental Mathematics a Computational Conclusion

Dalhousie Distributed Research Institute and Virtual Environment

MAA Online
The Mathematical Association of America

MAA Short Course on Experimental Mathematics (San Antonio Jan 10-11, 2006)

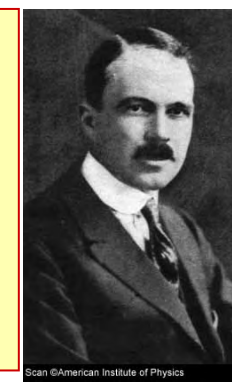
Jonathan Borwein, FRSC www.cs.dal.ca/~jborwein

Canada Research Chair in Collaborative Technology

"I feel so strongly about the wrongness of reading a lecture that my language may seem immoderate .... The spoken word and the written word are quite different arts .... I feel that to collect an audience and then read one's material is like inviting a friend to go for a walk and asking him not to mind if you go alongside him in your car."

Sir Lawrence Bragg

What would he say about .ppt?

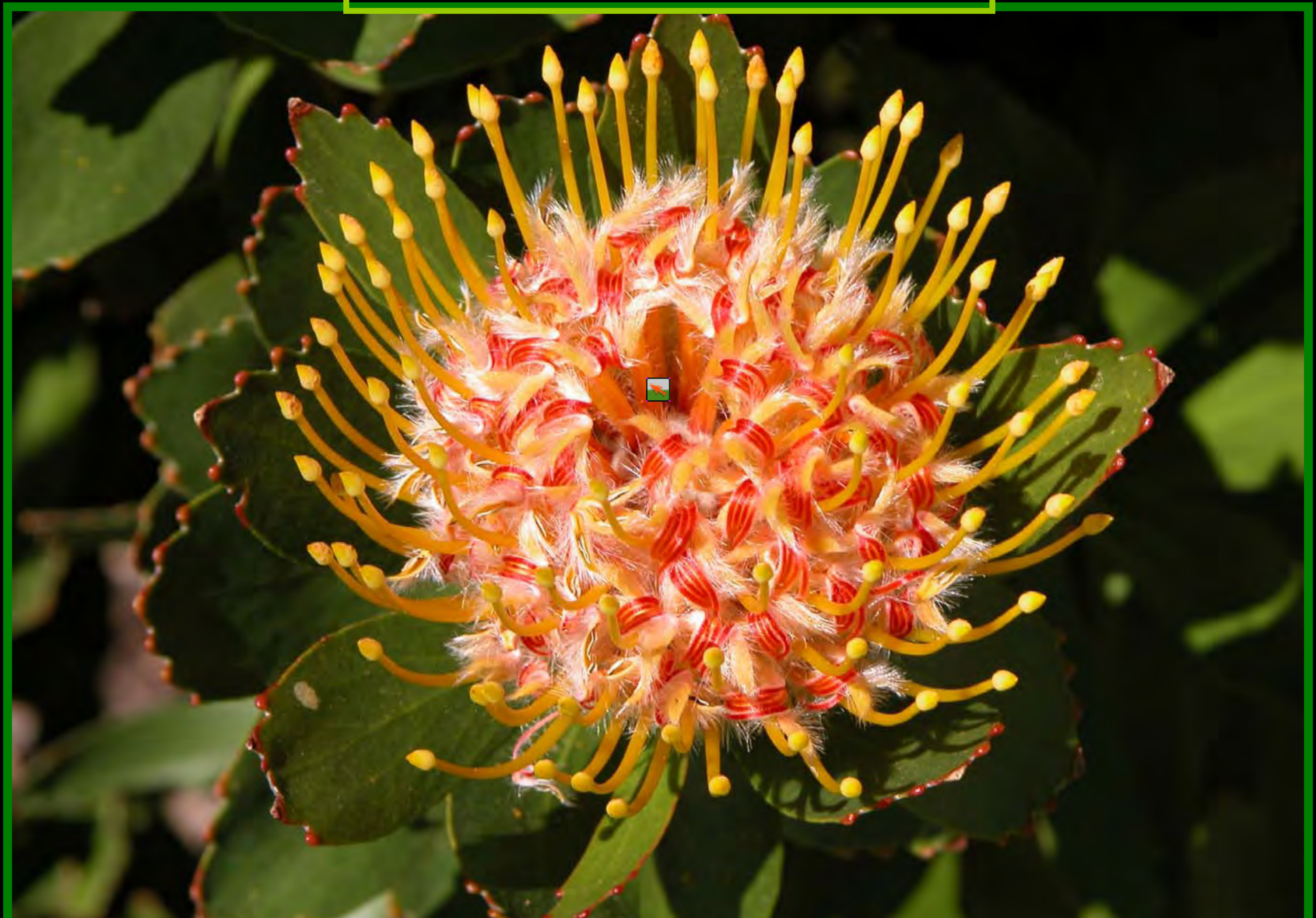


DALHOUSIE UNIVERSITY
Inspiring Minds

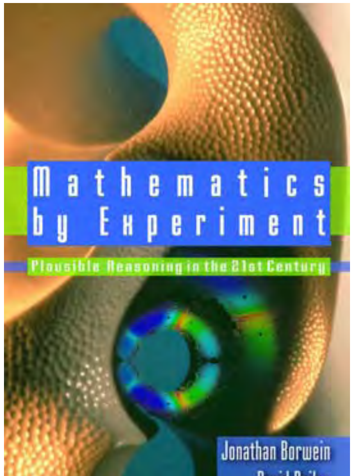
AARMS

Revised 02/01/06

San Antonio 2006

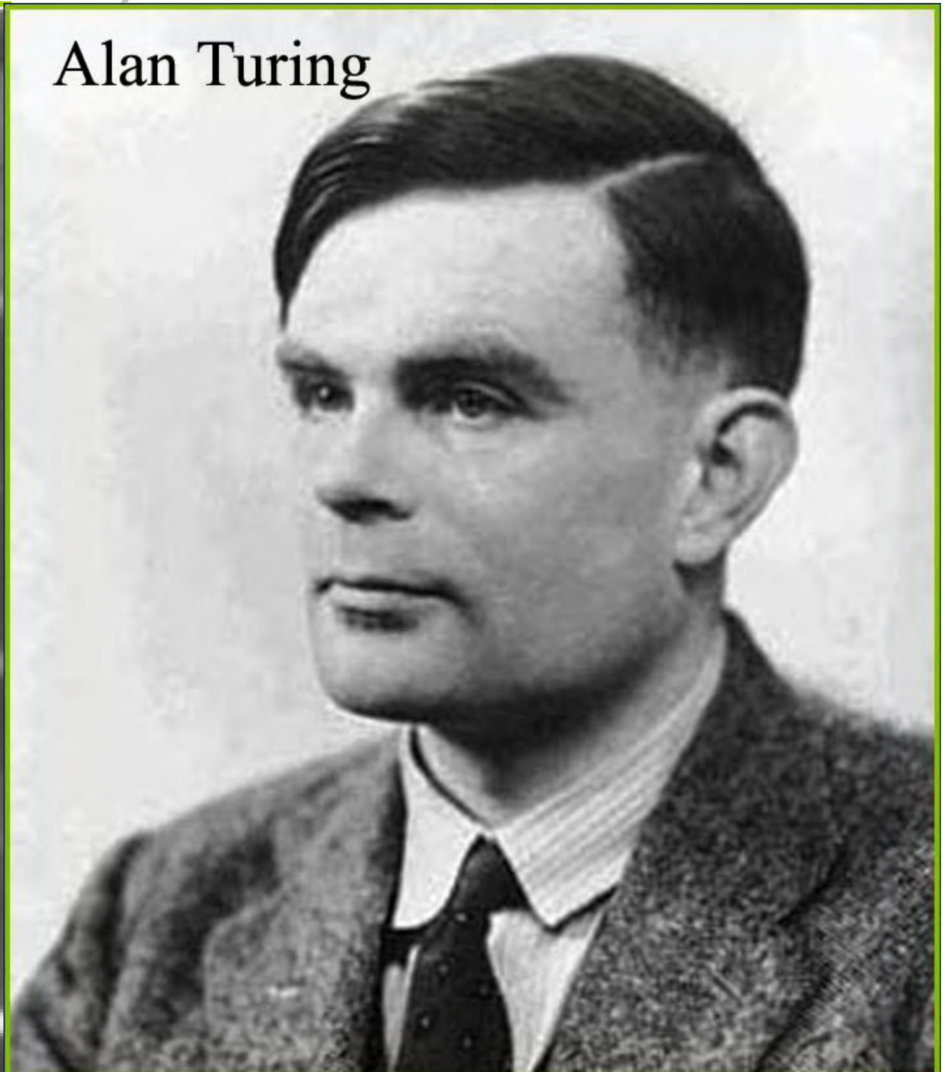


VIII. Experimental Mathematics a Computational Conclusion



ABSTRACT

"If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics." (**Kurt Godel,**1951)



We shall further explore various tools available for deciding what to believe in mathematics and, using accessible examples, illustrate the rich experimental tool-box mathematicians now have access to.

To explain how mathematicians may use **High Performance Computation** (HPC) and what we have in common with other computational scientists I shall include various **HPM** problems including:

$$\int_0^\infty \cos(2x) \prod_{n=1}^\infty \cos\left(\frac{x}{n}\right) dx \stackrel{?}{=} \frac{\pi}{8},$$

which is both numerically and symbolically quite challenging ....

From the Magazine | Innovators

It Doesn't Figure

The Omega Man

By MICHAEL D. LEMONICK

Chaitin's universal halting constant

Ω=0.0000001000000100001000000100001110
111001100100111100010010011100
(Calude)

From the Sep. 12, 2005 issue
Posted Sunday, Sep. 04, 2005

TIME. Online Edition

Over the past few decades, Gregory Chaitin, a mathematician at IBM's T.J. Watson Research Center in Yorktown Heights, N.Y., has been uncovering the distressing reality that much of higher math may be riddled with unprovable truths--that it's really a collection of random facts that are true for no particular reason. And rather than deducing those facts from simple principles, “I'm making the suggestion that mathematics is done more like physics in that you come about things experimentally,” he says. “This will still be controversial when I'm dead. It's a majo do mathematics”

Chaitin's idea centers on a numb overed in 1975 and which is much too complicated to explain here. (Ch est for Omega, out this month, should help make omega clear.) Suffice s two major discoveries of 20th century math: Gödel's incompleteness th ays be unprovable statements in any system of math, and Turing's hal ssible to predict in advance whether a particular computer calculation ca

Sounds like a nonevent in be. Cryptographers assume that their mathematically based encryption s. "If any of these people wake up at night and worry," says Chaitin, "I' on."

True, but why ?

The first series below was proven by Ramanujan. The next two were proven by Computer (Wilf-Zeilberger).

The candidates:

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} r_3(n)\,(42n+5)\left(\frac{1}{4^3}\right)^n$$

$$\frac{8}{\pi^2} = \sum_{n=0}^{\infty} r_5(n)\,(20n^2+8n+1)\left(\frac{-1}{4}\right)^n$$

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} r_5(n)\,(820n^2+180n+13)\left(\frac{-1}{4^5}\right)^n$$

$$\boxed{\frac{32}{\pi^3} = \sum_{n=0}^{\infty} r_7(n)\,(168n^3+76n^2+14n+1)\left(\frac{1}{4^3}\right)^n}$$

Here, in terms of factorials and rising factorials:

$$r_N(n) := \frac{\binom{2n}{n}^N}{4^{nN}} = \left(\frac{(1/2)_n}{n!}\right)^N.$$

The 4th is only true

$$\boxed{r_N(n) \sim_n \frac{1}{n^{N/2}}}$$

"How extremely stupid not to have thought of that!"

Thomas Henry Huxley (1825-1895) known as \`Darwin's Bulldog' for his tireless defense of Darwin, was initially unconvinced of evolution. Converted by \`Origin of Species', he is recorded (much like Briggs meeting Napier) as saying

"*How extremely stupid not to have thought of that!*"

"All truths are easy to understand once they are discovered; the point is to discover them."

(Galileo Galilei, 1564-1642)

On the other hand

Galileo's view is not a view shared by all (Kuhn, Dewey, …). The following *thoughts on quantum theory by various scientists* come from the NYT of Dec 26, 2005.

- **"Those are the crazy people who are not working on quantum theory."** (Albert Einstein referring to the inmates of an insane asylum near his office in Prague, in 1911)

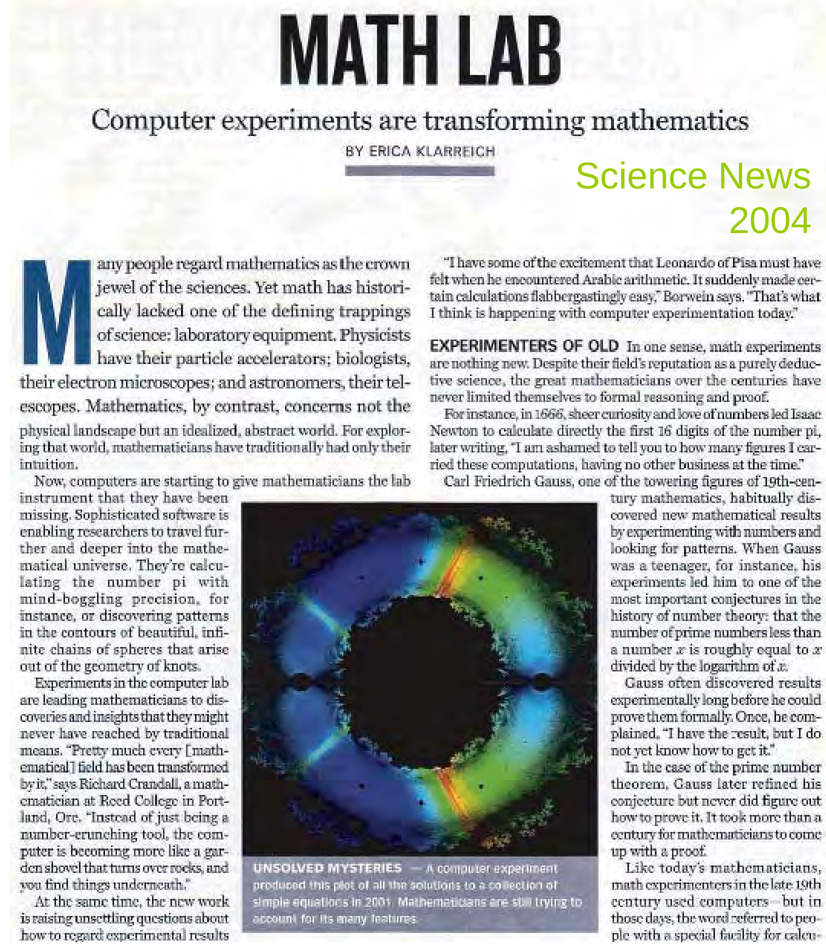
- **"Anyone who is not shocked by quantum theory has not understood a single word."** (Niels Bohr)

- **"I don't like it, and I'm sorry I ever had anything to do with it."** (Erwin Schrödinger about the probability interpretation of quantum mechanics)

"It says it's sick of doing things like inventories and payrolls, and it wants to make some break-throughs in astrophysics."

Experimental Mathodology (again)

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results



MATH LAB

Computer experiments are transforming mathematics

BY ERICA KLARREICH

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results

UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features.

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

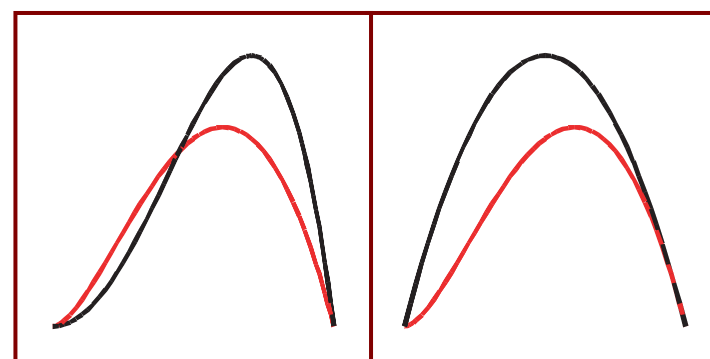
Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x.

Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

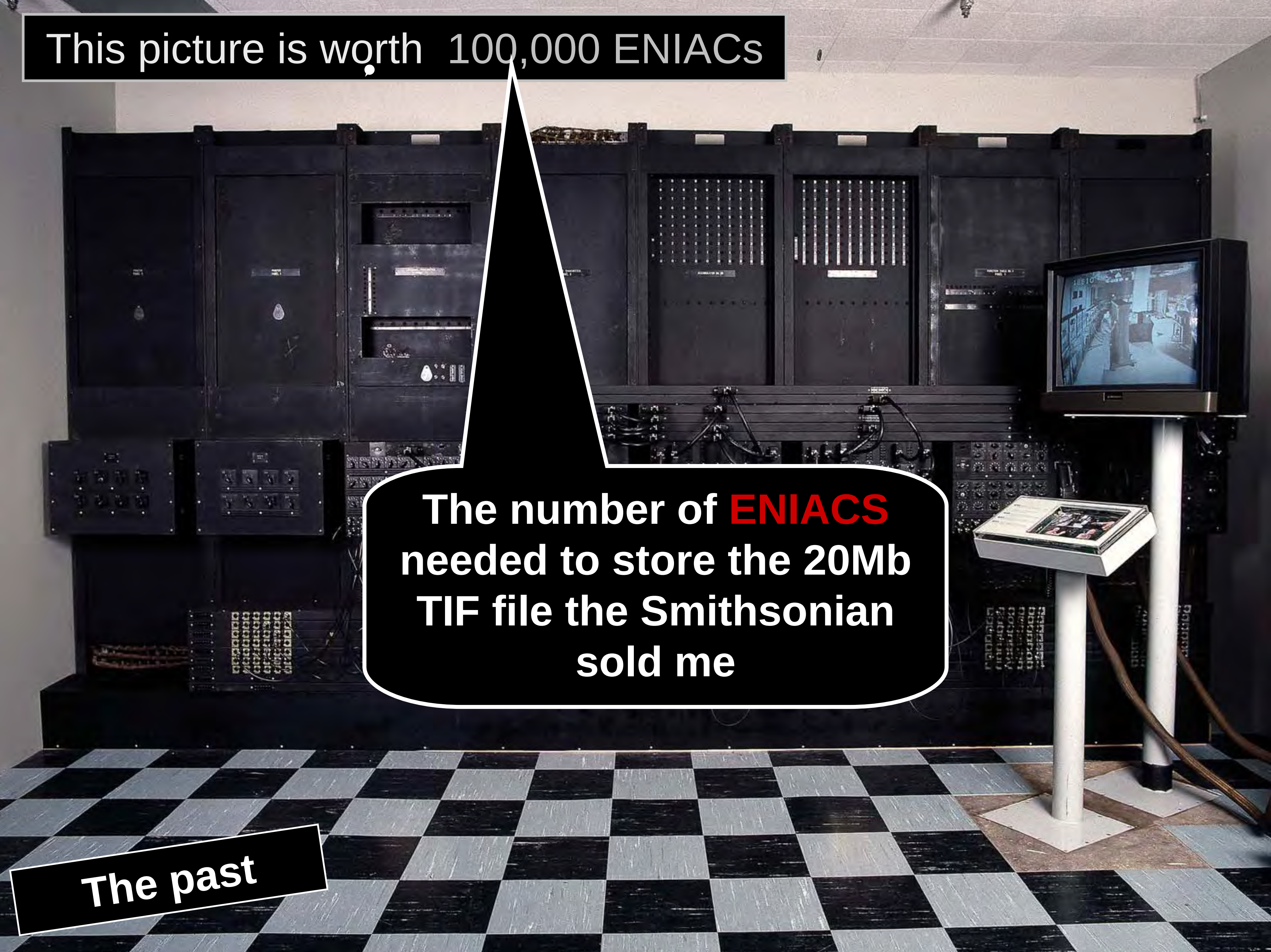
In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-

Science News 2004

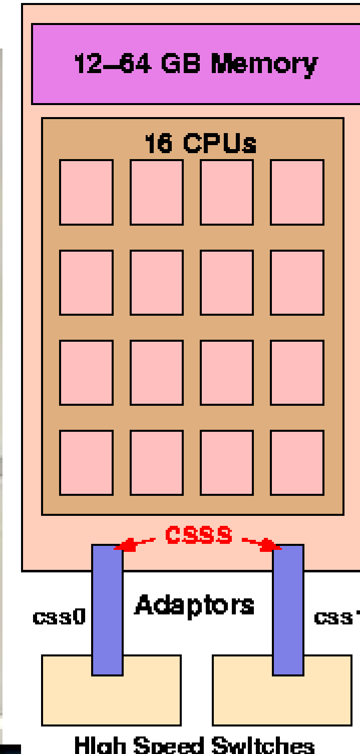


Comparing $-y^2\ln(y)$ (red) to $y-y^2$ and y^2-y^4


This picture is worth 100,000 ENIACs
The number of ENIACS needed to store the 20Mb TIF file the Smithsonian sold me
The past

NERSC's 6000 cpu Seaborg in 2004 (10Tflops/sec)

- we need new software paradigms for `bigga-scale' hardware



The present

Mathematical Immersive Reality
in Vancouver

IBM BlueGene/L system at LLNL

System
(64 cabinets, 64x32x32)

Supercomputer doubles own record

The Blue Gene/L supercomputer has broken its own record to achieve more than double the number of calculations it can do a second.

It reached 280.6 teraflops - that is 280.6 trillion calculations a second.

Blue Gene/L is the fastest computer in the world

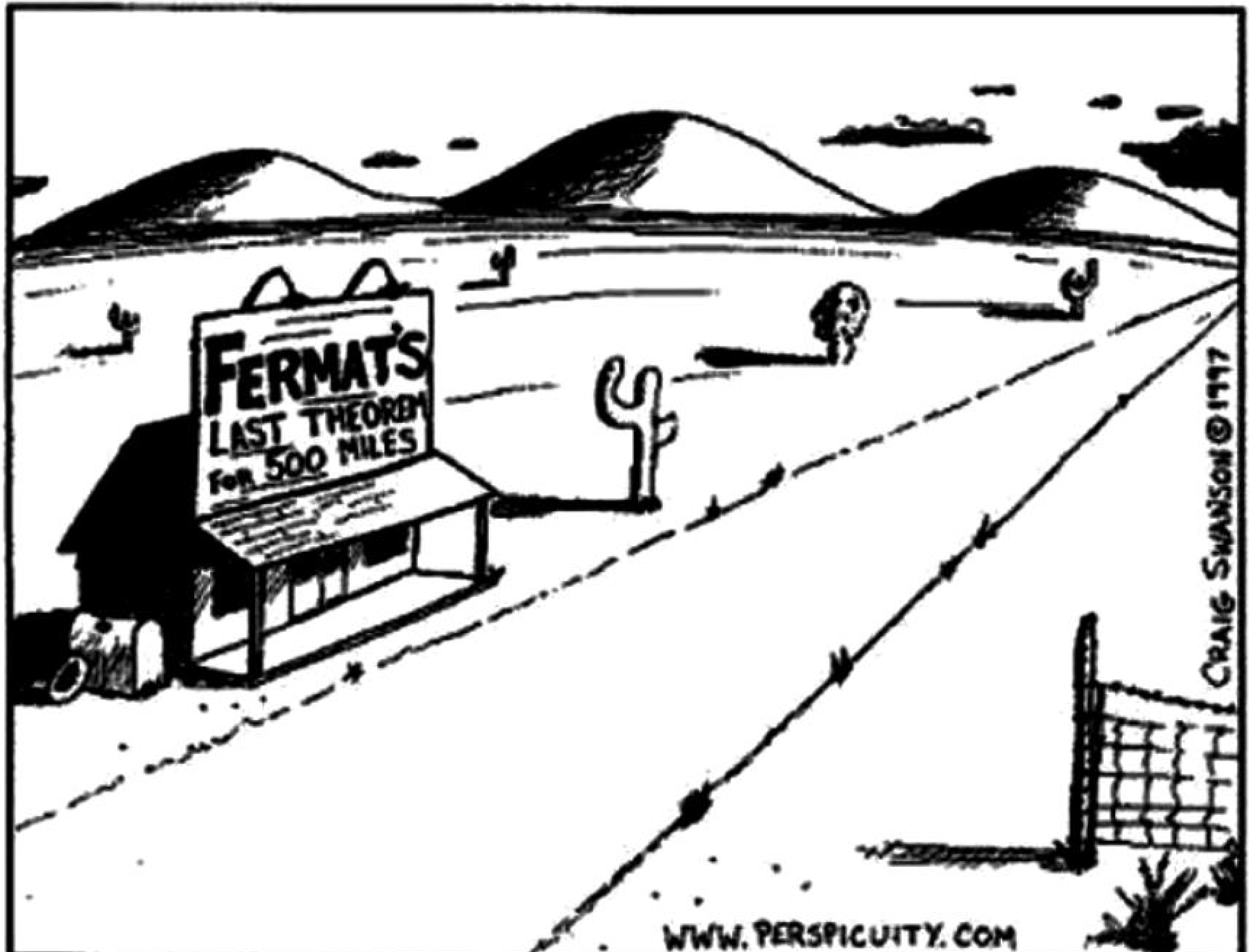
2.8/5.6 GF/s
4 MB

5.6/11.2 GF/s
0.5 GB DDR

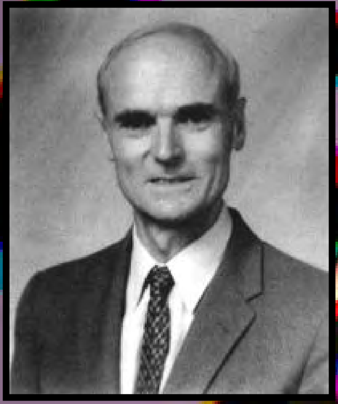
2^{17} cpu's

Oct 2005 It has now run Linpack benchmark at over **280 Tflop /sec** **(4 x Canadian-REN)**

The future


FERMAT'S
LAST THEOREM
FOR 500 MILES
CRAIG SWANSON © 1997
WWW.PERSPICUITY.COM

COXETER'S (1927) Kaleidescope

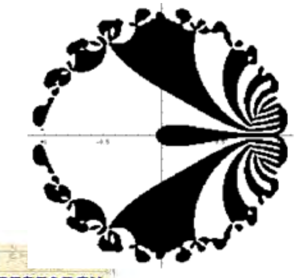


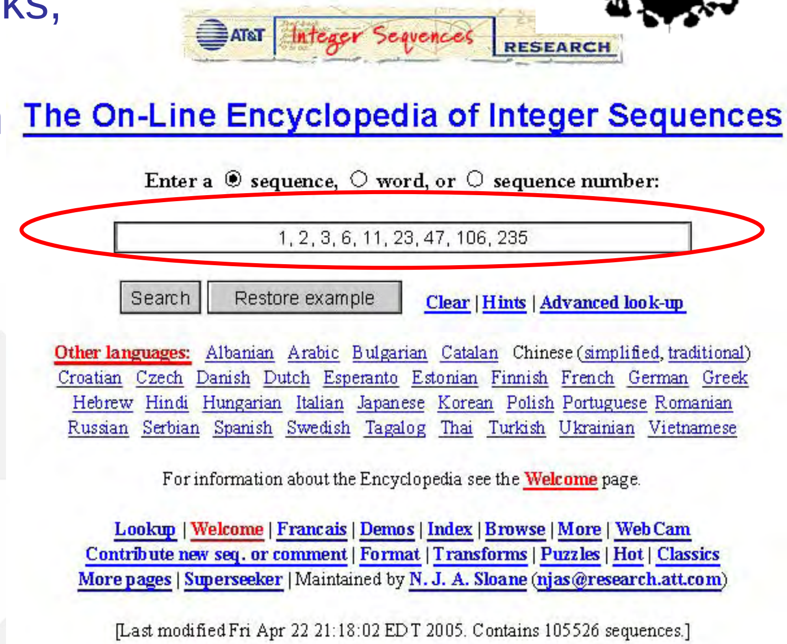
"What I appreciate even more than its remarkable speed and accuracy are the words of understanding and compassion I get from it."

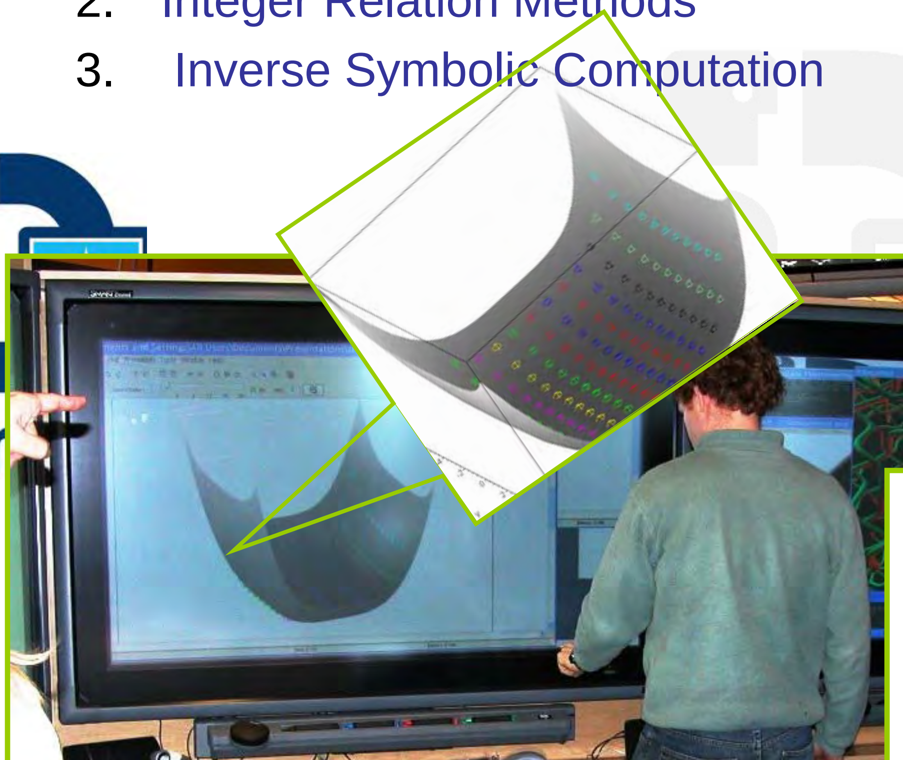
Numeric and Symbolic Computation

<u>…is central to my work</u> - with Dave Bailey -
meshed with visualization, randomized checks,
many web interfaces and

1. Massive (serial) Symbolic Computation
   - Automatic differentiation code
2. Integer Relation Methods
3. Inverse Symbolic Computation





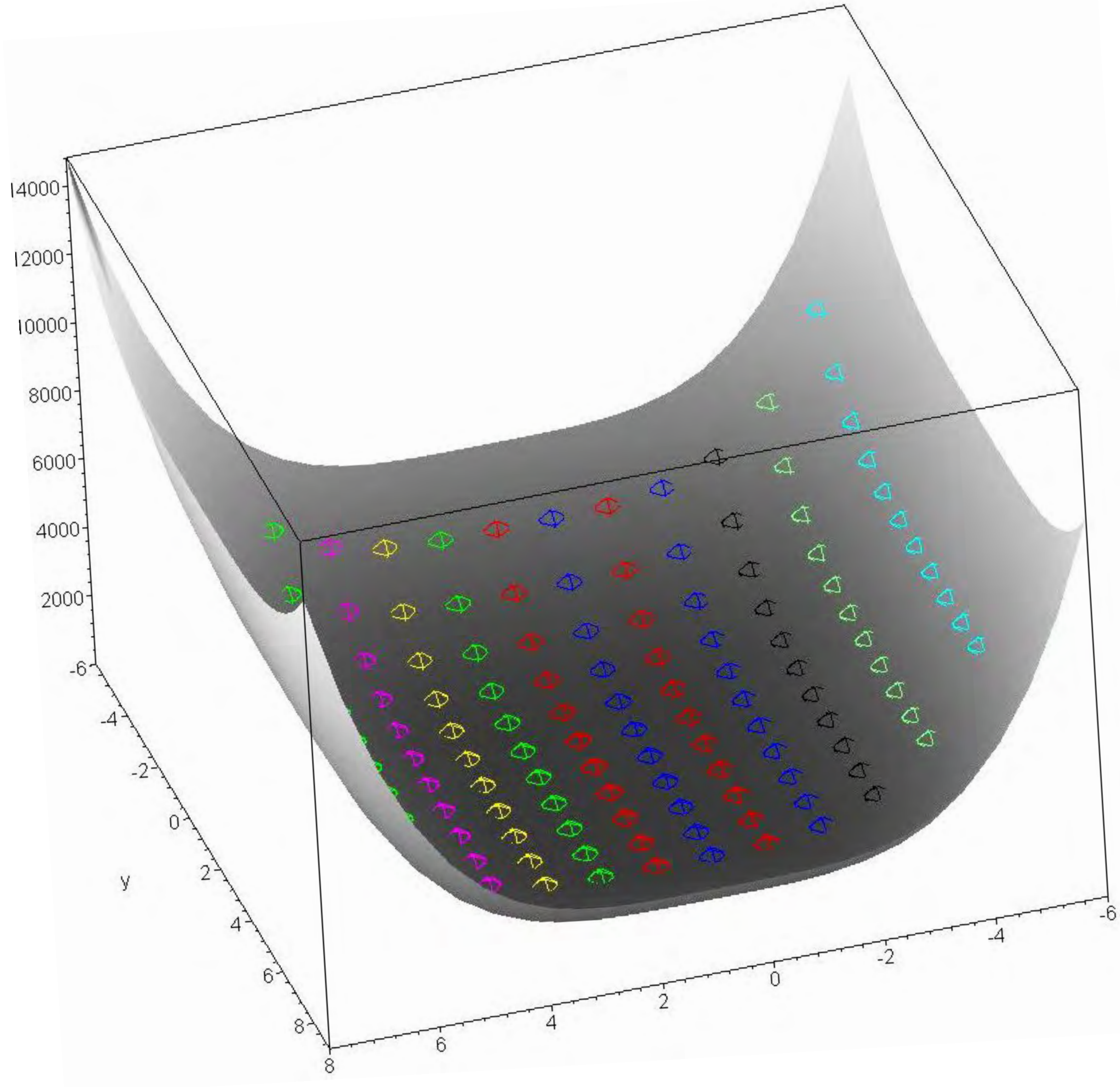


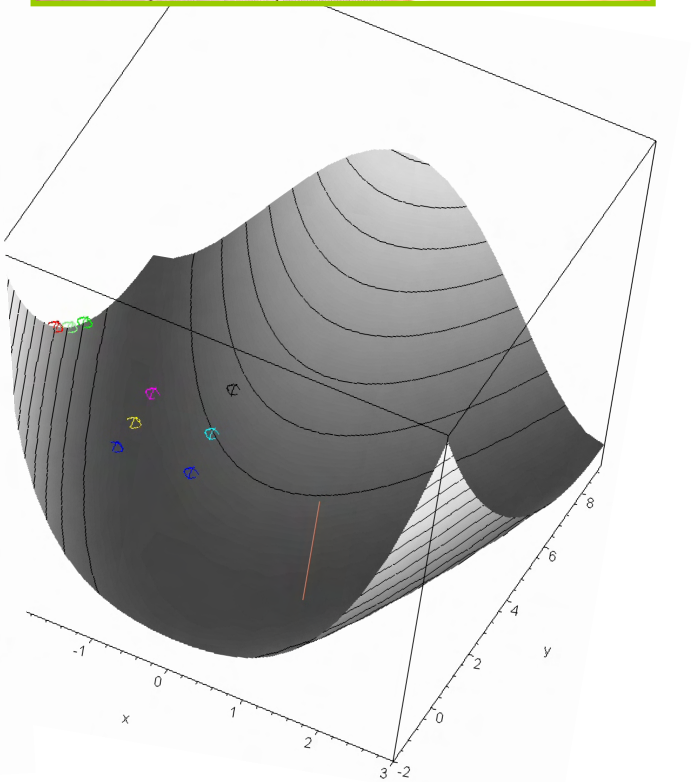
*Parallel derivative free optimization in **Maple***

<u>Other useful tools</u> : Parallel Maple
- Sloane's online sequence database
- Salvy and Zimmerman's generating function package '*gfun*'
- Automatic identity proving: Wilf-Zeilberger method for hypergeometric functions

Maple on SFU 192 cpu `bugaboo' cluster

- different node sets are in different colors





Greetings from the On-Line Encyclopedia of Integer Sequences!

Matches (up to a limit of 30) found for 1 2 3 6 11 23 47 106 235 :
[It may take a few minutes to search the whole database, depending on how many matches are found (the second and later look are faster)]

An Exemplary Database

ID Number: A000055 (Formerly M0791 and N0299)
URL: http://www.research.att.com/projects/OEIS?Anum=A000055
Sequence: 1,1,1,1,2,3,6,11,23,47,106,235,551,1301,3159,7741,19320, 48629,123867,317955,823065,2144505,5623756,14828074, 39299897,104636890,279793450,751065460,2023443032, 5469566585,14830871802,40330829030,109972410221
Name: Number of trees with n unlabeled nodes.
Comments: Also, number of unlabeled 2-gonal 2-trees with n 2-gons.
References F. Bergeron, G. Labelle and P. Leroux, Combinatorial Species and Tree-Like Structures, Camb. 1998, p. 279.
N. L. Biggs et al., Graph Theory 1736-1936, Oxford, 1976, p. 49.
S. R. Finch, Mathematical Constants, Cambridge, 2003, pp. 295-316.
D. D. Grant, The stability index of graphs, pp. 29-52 of Combinatorial Mathematics (Proceedings 2nd Australian Conf.), Lect. Notes Math. 403, 1974.
F. Harary, Graph Theory. Addison-Wesley, Reading, MA, 1969, p. 232.
F. Harary and E. M. Palmer, Graphical Enumeration, Academic Press, NY, 1973, p. 58 and 244.
D. E. Knuth, Fundamental Algorithms, 3d Ed. 1997, pp. 386-88.
R. C. Read and R. J. Wilson, An Atlas of Graphs, Oxford, 1998.
J. Riordan, An Introduction to Combinatorial Analysis, Wiley, 1958, p. 138.

Integrated real time use

- moderated
- 100,000 entries
- grows daily
- AP book had 5,000

Links: P. J. Cameron, Sequences realized by oligomorphic permutation groups J. Integ. Seqs. Vo
Steven Finch, Otter's Tree Enumeration Constants
E. M. Rains and N. J. A. Sloane, On Cayley's Enumeration of Alkanes (or 4-Valent Trees),
N. J. A. Sloane, Illustration of initial terms
E. W. Weisstein, Link to a section of The World of Mathematics.
Index entries for sequences related to trees
Index entries for "core" sequences
G. Labelle, C. Lamathe and P. Leroux, Labeled and unlabeled enumeration of k-gonal 2-tr

Formula: G.f.: A(x) = 1 + T(x)-T^2(x)/2+T(x^2)/2, where T(x) = x + x^2 + 2*x^3 + ...

A WARMUP Computational Proof

Suppose we know that 1<N<10 and that N is an integer

- **computing N to 1 significant place** **with a certificate** will prove the value of N. *Euclid's method* is basic to such ideas.

Likewise, suppose we <u>know</u> α is *algebraic of degree d and length λ*

(coefficient sum in absolute value)

If P is polynomial of degree D & length L **EITHER** $P(\alpha) = 0$ **OR**

$$|P(\alpha)| \geq \frac{1}{L^{d-1}l^{D}}$$

Example (MAA, April 2005). Prove that

$$\int_{-\infty}^{\infty} \frac{y^2}{1+4\,y+y^6-2\,y^4-4\,y^3+2\,y^5+3\,y^2}\,dy = \pi$$

Proof. Purely **qualitative analysis** with partial fractions and arctans shows the integral is π β where β is algebraic of degree *much* less than **100 (actually 6)**, length *much* less than **100,000,000**.With **P(x)=x-1** (D=1,L=2, d=6, λ=?), this means *checking* the identity to **100** places is plenty of **PROOF.**

A fully symbolic Maple proof followed. **QED** $|\beta - 1| < 1/(32\lambda) \mapsto \beta = 1$

Hilbert and Witten

We explore a surprising relation between the **Witten Zeta function** for r, s>1/2

$$\mathcal{W}(r,s,t) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^r\, m^s\,(n+m)^t}$$

and **Hilbert's inequality** for positive sequences and 1/p+1/q = ∞

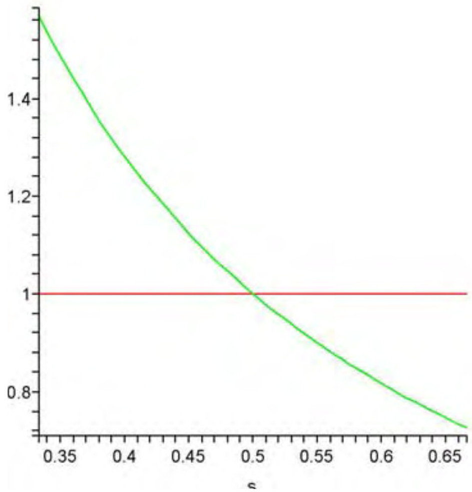
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n\, b_m}{n+m}$$

$$< \pi \csc\left(\frac{\pi}{p}\right) \|a_n\|_p \, \|b_n\|_q.$$

The Best "Hilbert Constant"

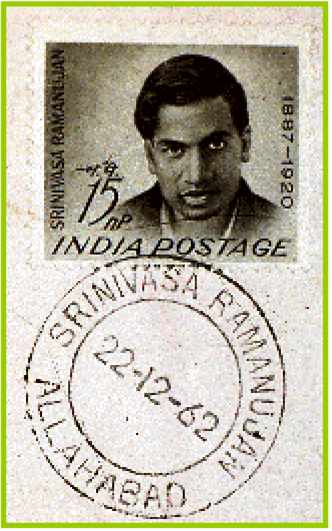
is the one given. For p=2, we (with effort) numerically explore

$$\mathcal{R}(s) := \frac{\mathcal{W}(s,s,1)}{\pi\,\zeta(2s)}$$

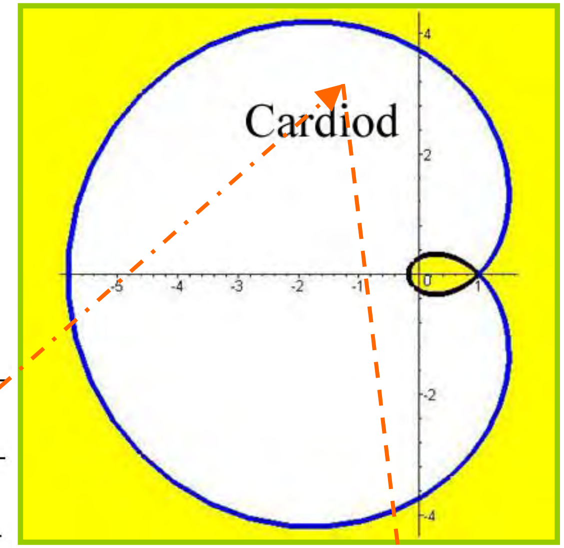


and so determine $\lim_{s\to 1/2}\mathcal{R}(s) = 1.$

This confirms that the constant is best possible and the method works more generally – with many open questions!



Ramanujan's Arithmetic-Geometric Continued fraction (CF)



$$R_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ldots}}}}$$

For a,b>0 the CF satisfies a lovely symmetrization

$$\mathcal{R}_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_\eta(a,b) + \mathcal{R}_\eta(b,a)}{2}$$

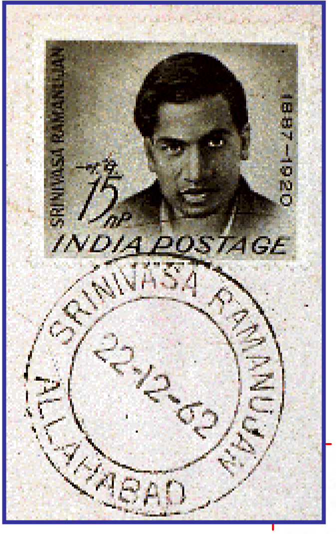
Computing directly was too hard; **even 4 places of** $\mathcal{R}_1(1,1) = \log 2$?

We wished to know for which a/b in C this all held

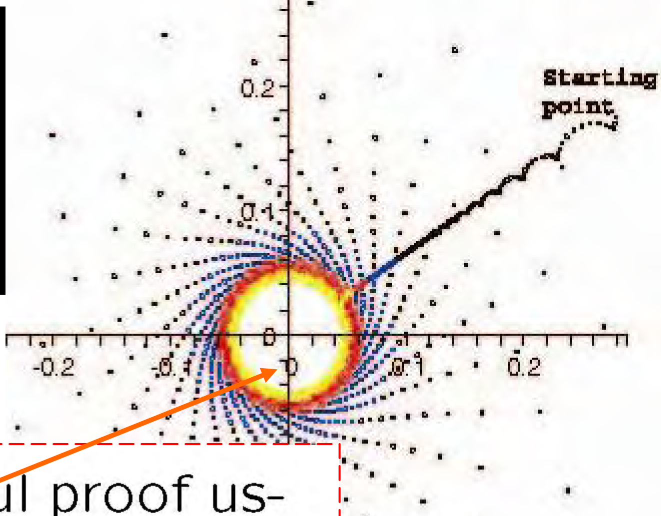
The scatterplot revealed a precise cardioid where r=a/b.

- **which discovery it remained to prove?**

$$\left|\frac{a+b}{2}\right| \geq \sqrt{|ab|}$$



Ramanujan's Arithmetic-Geometric Continued fraction



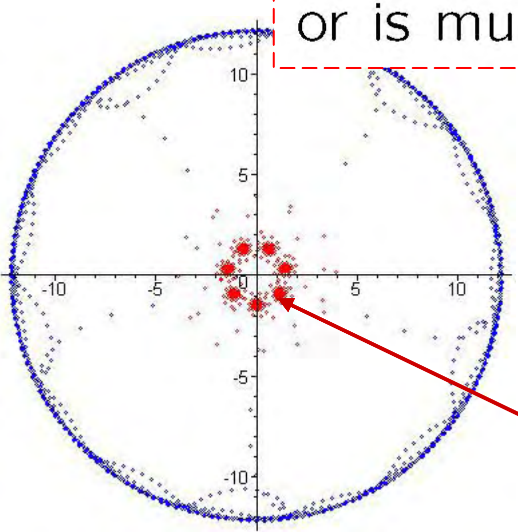
1. The Blackbox

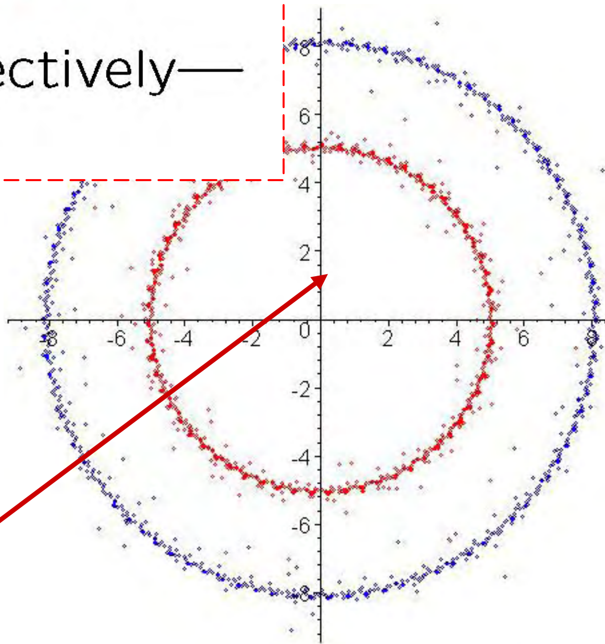
2. Seeing convergence

Six months later we had a beautiful proof using genuinely new dynamical results. Starting from the *dynamical system* $t_0 := t_1 := 1$:

$$t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for n even, odd respectively— or is much more general.*

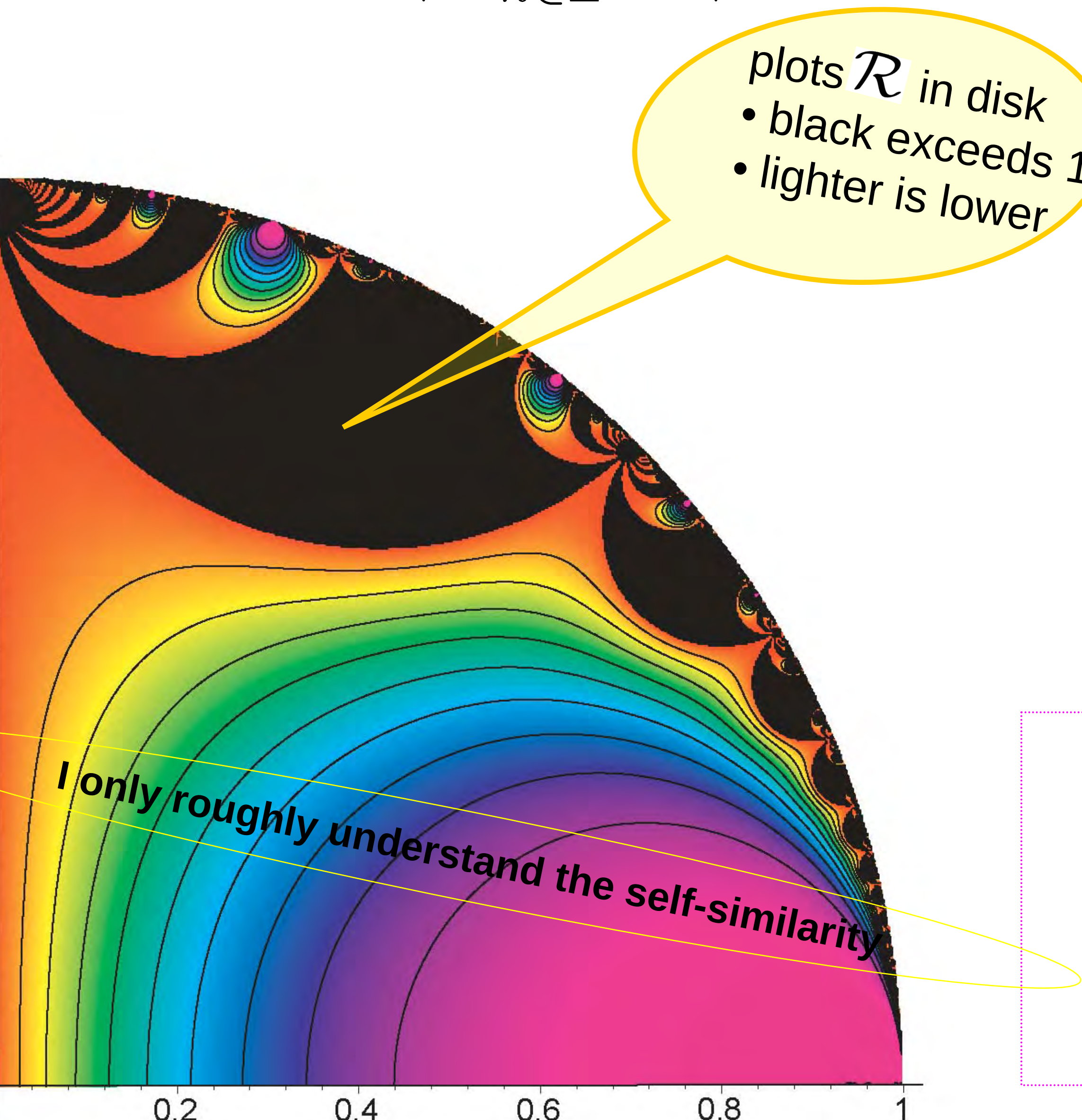


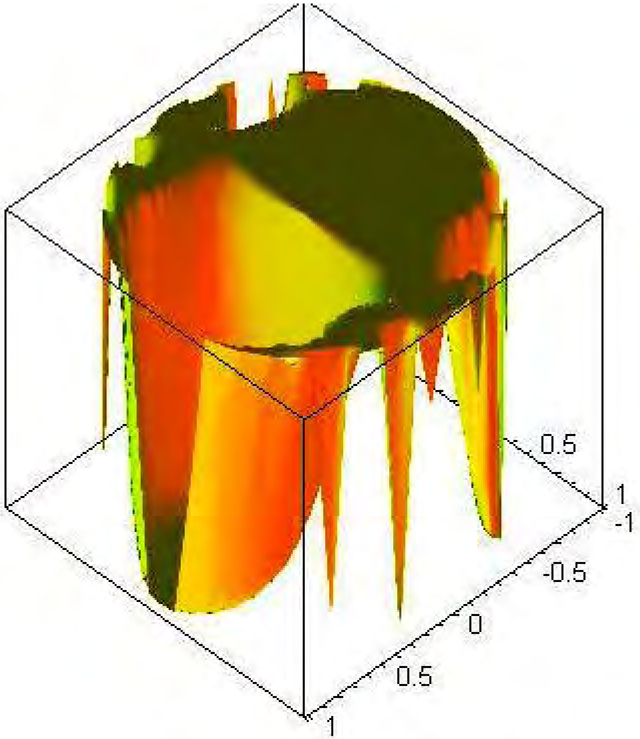


3. Attractors. *Normalizing* by $n^{1/2}$ three cases appear

$$\mathcal{R} = \frac{|\sum_{n\in\mathbf{Z}}(-1)^n q^{n^2}|}{|\sum_{n\in\mathbf{Z}} q^{n^2}|}$$

FRACTAL of a Modular Inequality



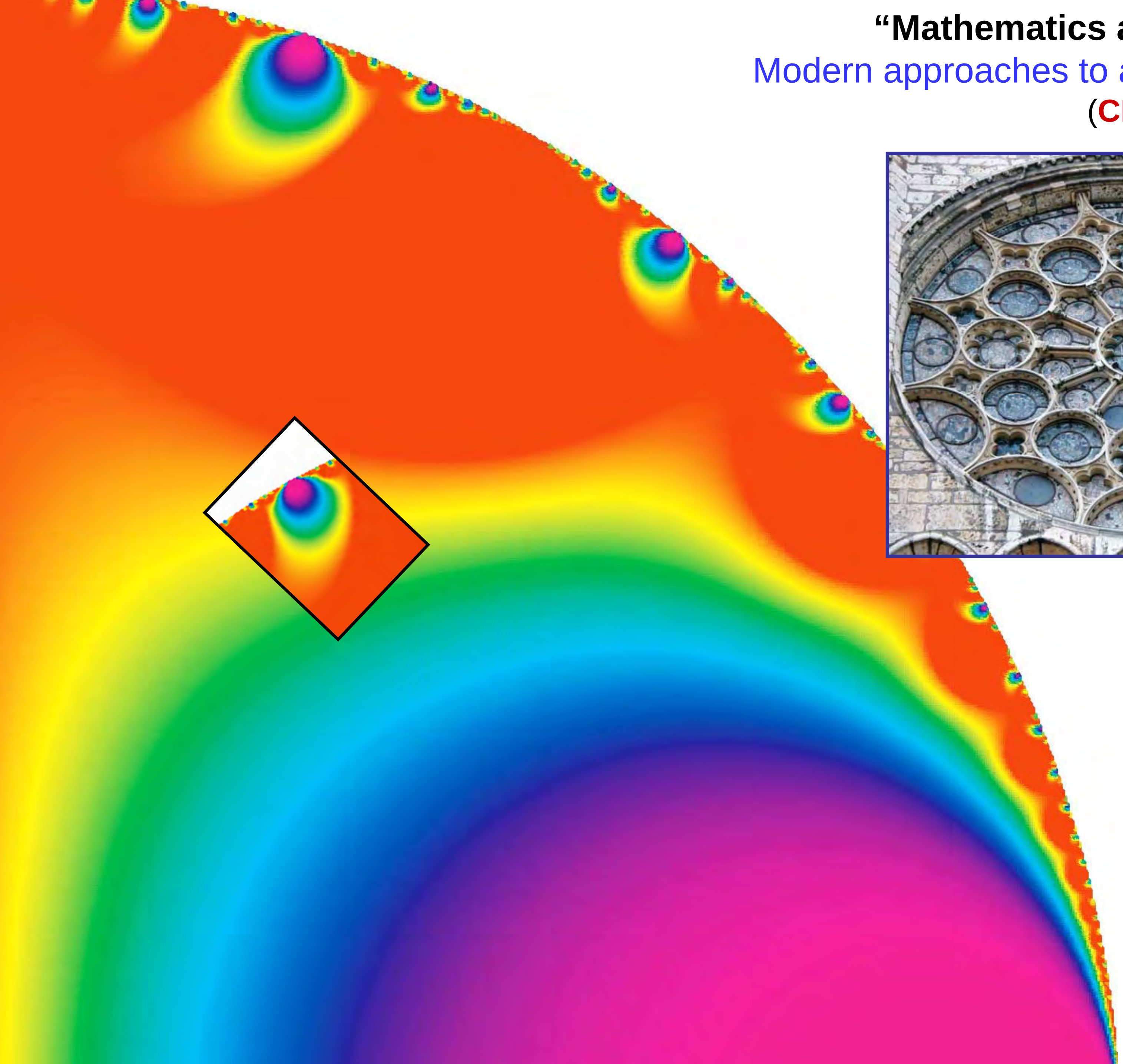


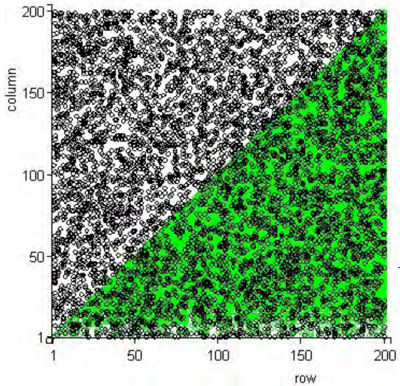
- ✓ related to Ramanujan's continued fraction
- ✓ took several hours to print
- ✓ Crandall/Apple has parallel print mode

“Mathematics and the aesthetic

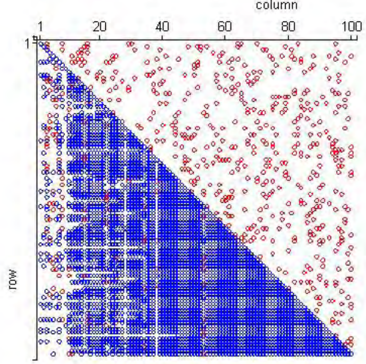
Modern approaches to an ancient affinity”

(**CMS-Springer**, 2006)





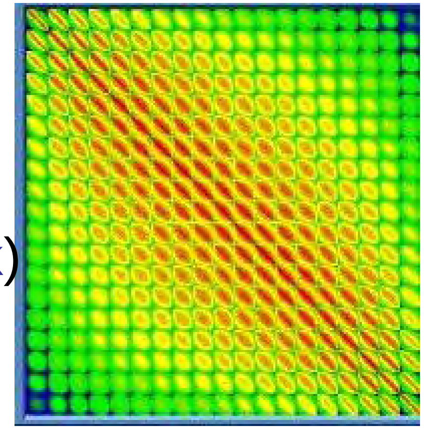
Pseudospectra or Stabilizing Eigenvalues



Gaussian elimination of random sparse (10%-15%) matrices

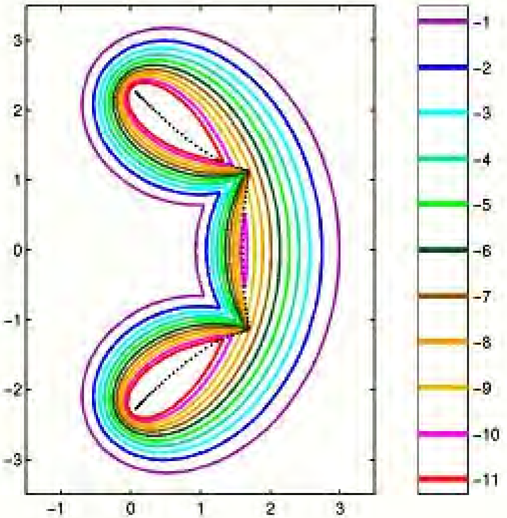
'Large' (10^5 to 10^8) **Matrices must be seen**

A dense inverse



1. conditioning and ill-conditioning
2. sparsity and its preservation
3. eigenvalues
4. singular values (helping Google work)

Pseudospectrum of a banded matrix

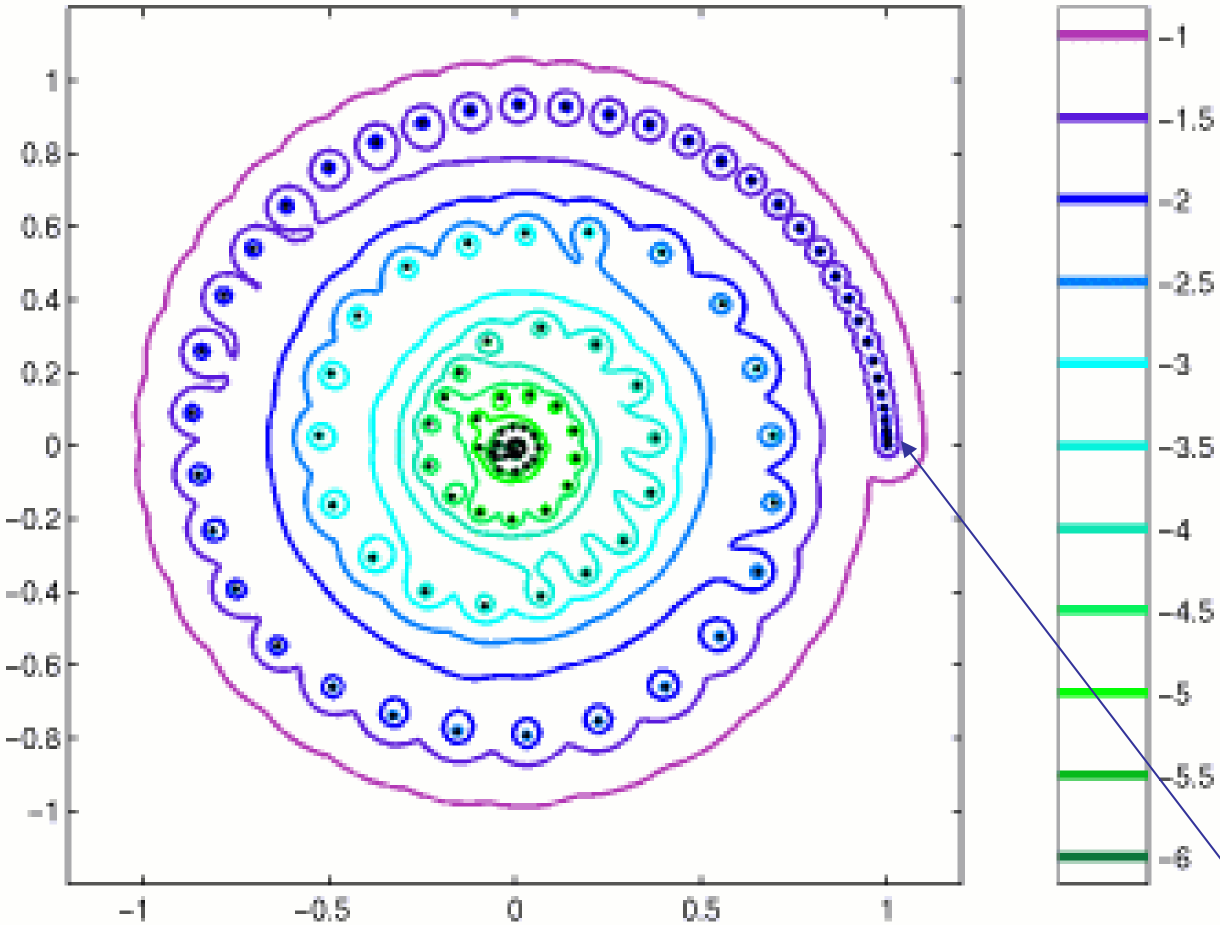


The pseudo spectrum of A: for $\varepsilon > 0$

$$\sigma_\varepsilon(A) = \{\lambda : \inf \|Ax - \lambda x\| \leq \varepsilon\}$$

http://web.comlab.ox.ac.uk/projects/pseudospectra

An Early Use of Pseudospectra (Landau, 1977)



$\varepsilon = 10^{-t}$

- An infinite dimensional integral equation in laser theory
  - discretized to a matrix of dimension **600**
  - projected onto a well chosen invariant subspace of dimension **109**

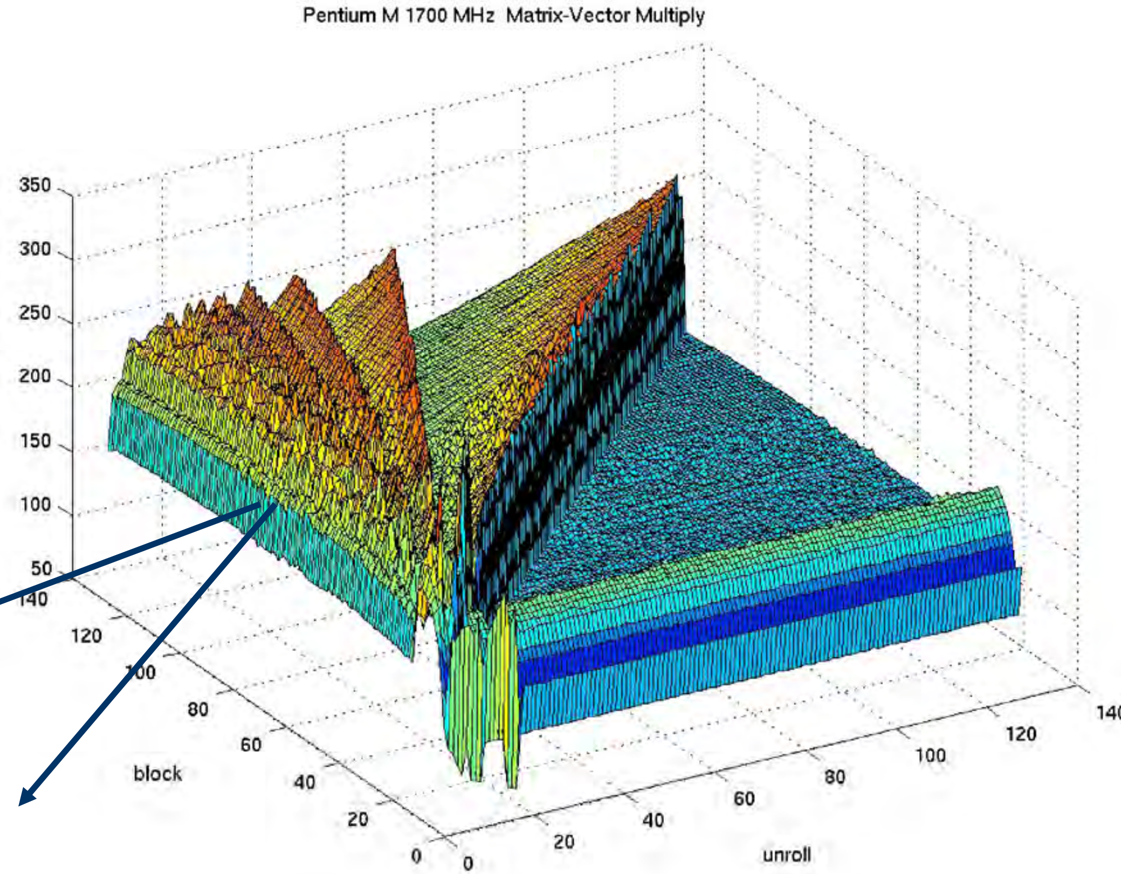
Experimentation with DGEMV (matrix-vector multiply):

128x128=16,384 cases.

Experiment took 30+ hours to run.

Best performance = 338 Mflop/s with blocking=11 unrolling=11

Original performance = 232 Mflop/s



Visual Representation of Automatic Code Parallelization

Fast Arithmetic
(Complexity Reduction in Action)

Multiplication

- Karatsuba multiplication (200 digits +) or Fast Fourier Transform (FFT)

… in ranges from 100 to 1,000,000,000,000 digits

- The other operations via Newton's method $\times, \div, \sqrt{\cdot}$
- Elementary and special functions via Elliptic integrals and Gauss AGM

$$O\left(n^{\log_2(3)}\right)$$

For example:

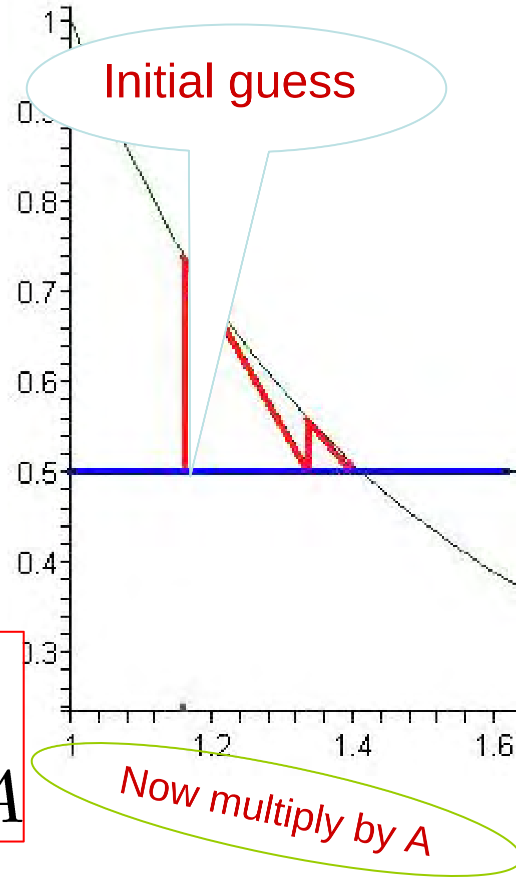
$$\begin{aligned}
&\left(a + c \cdot 10^N\right) \times \left(b + d \cdot 10^N\right) \\
= \;& ab + (ad + bc) \cdot 10^N + cd \cdot 10^{2N} \\
= \;& ab + \underbrace{\{(a+c)(b+d) - ab - cd\}}_{\text{three multiplications}} \cdot 10^N + cd \cdot 10^{2N}
\end{aligned}$$

Karatsuba replaces one 'times' by many 'plus'

FFT multiplication of multi-billion digit numbers reduces centuries to minutes. Trillions must be done with Karatsuba!

Newton's Method for Elementary Operations and Functions

$$x \leftarrow x - \frac{f(x)}{\frac{d}{dx}f(x)}$$

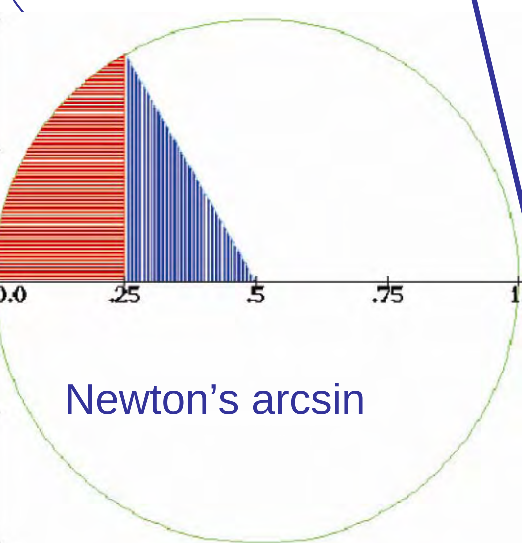


1. Doubles precision at each step
   Newton is **self correcting** and **quadratically convergent**
2. Consequences for **work needed**:

$$x \leftarrow x(2 - xA)$$

$$x \leftarrow 1/2\,x\left(3 - x^2A\right)$$

$$\boxed{\begin{array}{l} '\div = 4\times' : 1/x = A \\ '\sqrt{\cdot} = 6\times' : 1/x^2 = A \end{array}}$$



3. For the **logarithm** we approximate by **elliptic integrals (AGM)** which admit **quadratic transformations**: near zero

$$\boxed{\frac{d}{dk}K(k) \sim \log\left(\frac{4}{k}\right)}$$

4. We use **Newton** to obtain the **complex exponential**

So ***all*** **elementary functions** are fast computable

"What it comes down to is our software
is too hard and our hardware is too soft."

Peter Borwein
in front of
Helaman Ferguson's
work

CMS Meeting
December 2003
SFU Harbour Centre

Ferguson uses high tech tools and micro engineering at NIST to build monumental math sculptures

PSLQ and Zeta

Euler (1707-73)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

1. via PSLQ to 50,000 digits
(250 terms)

$$= \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \ldots$$

2005 Bailey, Bradley & JMB *discovered and proved* - in Maple - three *equivalent* binomial identities

1

$$\mathcal{Z}(x) = 3 \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)} \prod_{n=1}^{k-1} \frac{4x^2 - n^2}{x^2 - n^2}$$

$$= \sum_{k=0}^{\infty} \zeta(2k+2) x^{2k} = \sum_{n=1}^{\infty} \frac{1}{n^2 - x^2}$$

$$= \frac{1 - \pi x \cot(\pi x)}{2x^2}$$

2

2. reduced as hoped

$$3n^2 \sum_{k=n+1}^{2n} \frac{\prod_{m=n+1}^{k-1} \frac{4n^2 - m^2}{n^2 - m^2}}{\binom{2k}{k}\left(k^2 - n^2\right)} = \frac{1}{\binom{2n}{n}} - \frac{1}{\binom{3n}{n}}$$

3

$${}_3F_2\left(\begin{matrix} 3n, n+1, -n \\ 2n+1, n+1/2 \end{matrix}; \frac{1}{4}\right) = \frac{\binom{2n}{n}}{\binom{3n}{n}}$$

3. was easily **computer proven**
(Wilf-Zeilberger)

Wilf-Zeilberger Algorithm

is a form of automated telescoping: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{n+1} \right\} = 1$

- ✓ **AMS Steele Research Prize** winner. In **Maple 9.5** set:

$$F := \frac{(3\,n+k-1)!\,(n+k)!\,(-n+k-1)!\,(2\,n)!\,(n-1/2)!\,(1/4)^k}{(3\,n-1)!\,n!\,(-n-1)!\,(2\,n+k)!\,(n-1/2+k)!\,k!}, \quad r := \frac{\binom{2\,n}{n}}{\binom{3\,n}{n}}$$

and execute:

```
>   with(SumTools[Hypergeometric]):
>   WZMethod(F,r,n,k,'certify'): certify;
```

which returns the certificate

```
          /     2                        \
          \11 n  + 1 + 6 n + k + 5 k n/ k
      - ---------------------------------
           3 (n - k + 1) (2 n + k + 1) n
```

This proves that summing $F(n,k)$ over k produces $r(n)$, as asserted.

If this were the philosophy talk I should discuss the following two quotes and further defend our philosophy of mathematics:

<u>Abstract of the future</u> *We show in a certain precise sense that the **Goldbach Conjecture** is true with probability larger than 0.99999 and that its complete truth could be determined with a budget of 10 billion.*

"Secure Mathematical Knowledge"

"It is a waste of money to get absolute certainty, unless the conjectured identity in question is known to imply the Riemann Hypothesis."

Doron Zeilberger, 1993

Goldbach: every even number (>2) is a sum of two primes?

So we will look at the now concrete **Riemann Hypothesis**

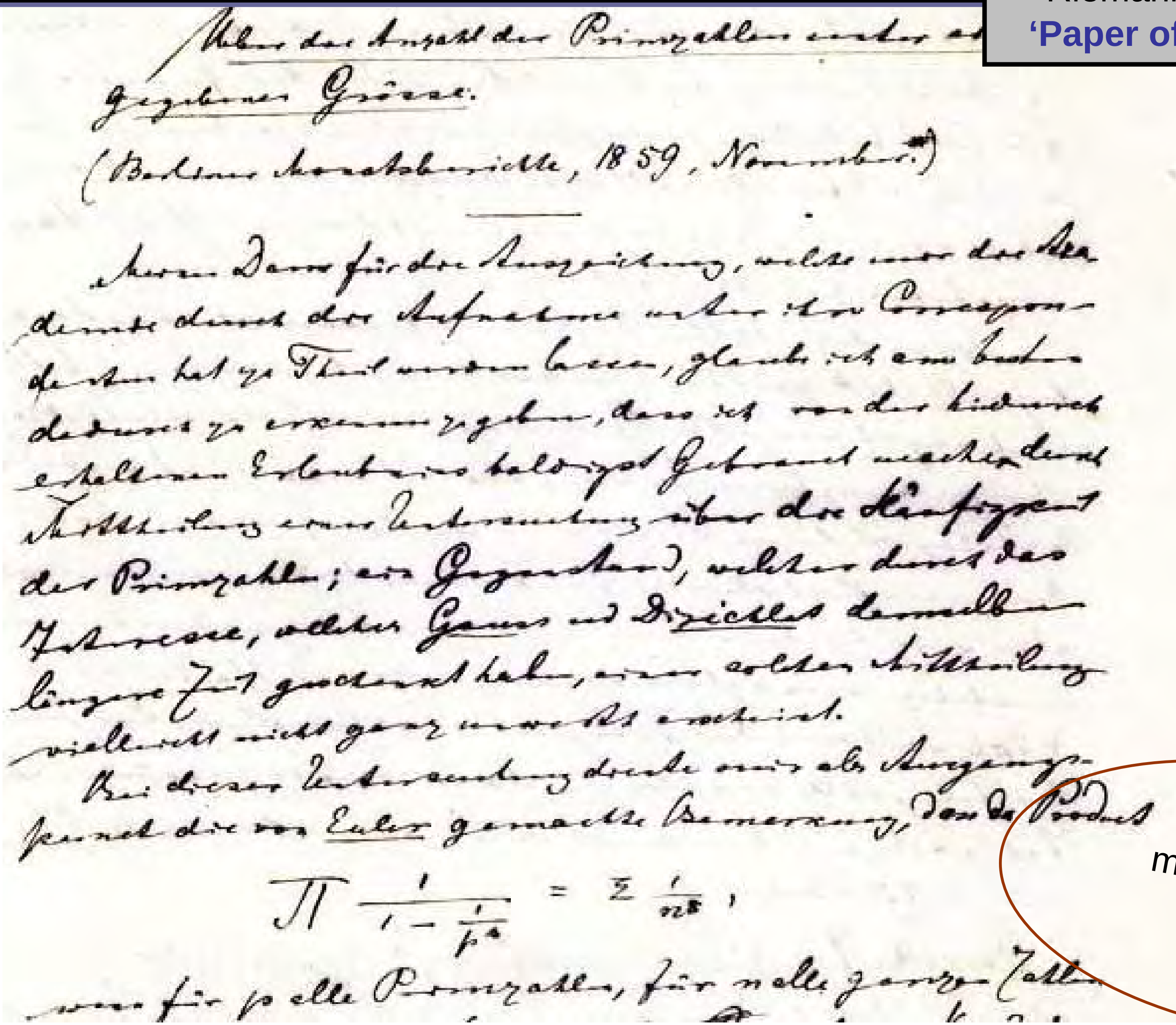
…

Uber die Anzahl der Primzahlen unter einer Gegebenen Grosse

On the number of primes less than a given quantity

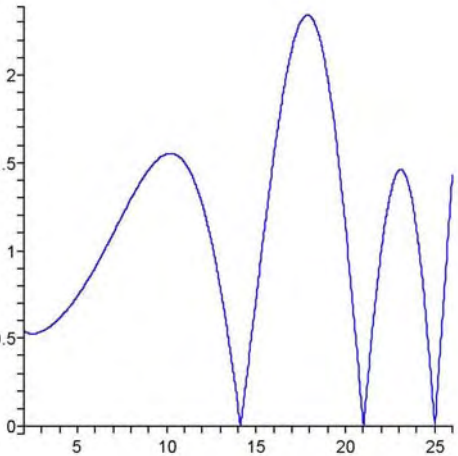
Riemann's six page 1859

'Paper of the Millennium'?

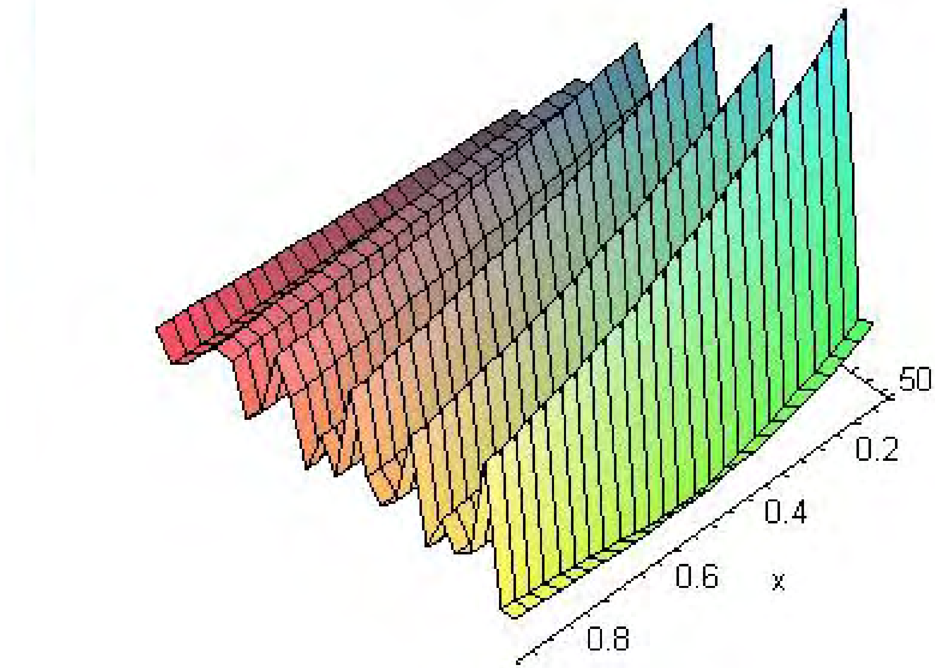


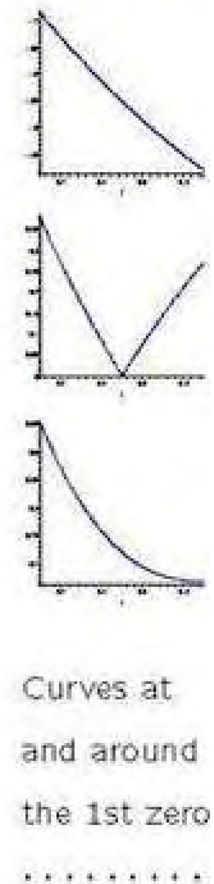
RH is so important because it yields precise results on distribution and behaviour of primes

Euler's product makes the key link between **primes and ζ**



The Modulus of Zeta and the **Riemann Hypothesis** (A Millennium Problem)





The imaginary parts of first 4 zeroes are:

14.134725142
21.022039639

25.010857580
30.424876126

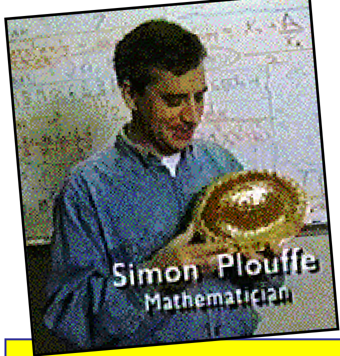
The first 1.5 billion are on the *critical line*

Yet at 10^{22} the ***"Law of small numbers"*** still rules (Odlyzko)

'All non-real zeros have real part one-half'

(The Riemann Hypothesis)

Note the **monotonicity** of **$x \mapsto |\zeta(x+iy)|$** is **equivalent to RH** discovered in a Calgary class in 2002 by Zvengrowski and Saidak



PSLQ and Hex Digits of Pi

Finalist for the $100K Edge of Computation Prize

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{k\,2^k}$$

My brother made the observation that this log formula allows one to compute binary digits of log 2 *without*

EdgeThe Third Culture

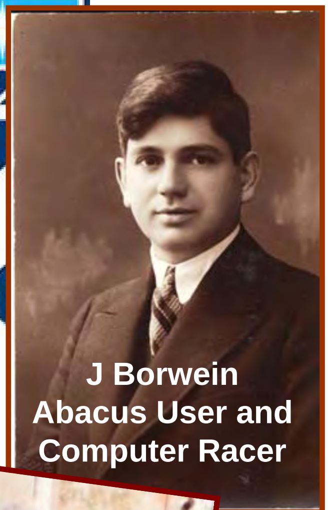
[Home](#) | [About Edge](#) | [Features](#) | [Edge Editions](#) | [Press](#) | [The Reality Club](#) | [Third Culture](#) | [Digerati](#) | [Edge Search](#)

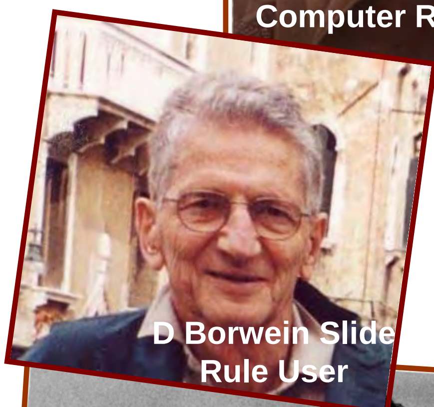
THE $100,000 EDGE OF COMPUTATION SCIENCE PRIZE

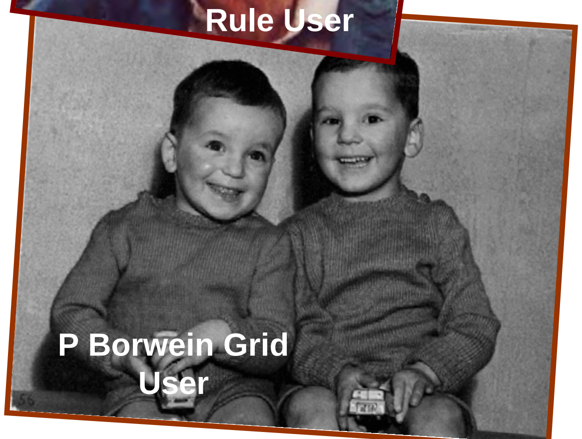
For individual scientific work, extending the computational idea, performed, published, or newly applied within the past ten years.

The Edge of Computation Science Prize, established by Edge Foundation, Inc., is a $100,000 prize initiated and funded by science philanthropist Jeffrey Epstein.

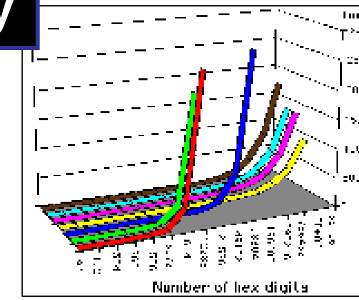
The pre-designed Algorithm ran the next day







ALGORITHMIC PROPERTIES



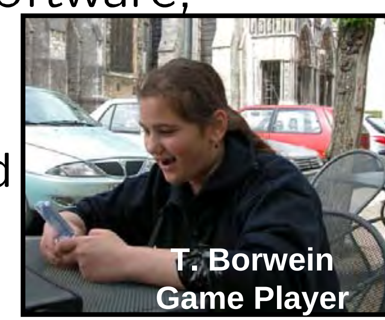
(1) produces a modest-length string hex or binary digits of π, beginning at an arbitrary position, using no prior bits;

Now built into some compilers!

(2) is implementable on any modern computer;

(3) requires no multiple precision software;

(4) requires very little memory; and



(5) has a computational cost growing only slightly faster than the digit position.

PiHex– A distributed effort to calculate Pi

http://www.cecm.sfu.ca/projects/pihex/

Getting Started Latest Headlines www.icbc.ca

- Join PiHex
  - Download
  - Source Code
- About
  - Credits
  - Status
  - Top Producers
- What's New?
- Other Projects
- Who am I?
- Email me!

174962

hits since the counter last reset.

PiHex

A distributed effort to calculate Pi.

The Quadrillionth Bit of Pi is '0'!
The Forty Trillionth Bit of Pi is '0'!
The Five Trillionth Bit of Pi is '0'!

Percival 2004

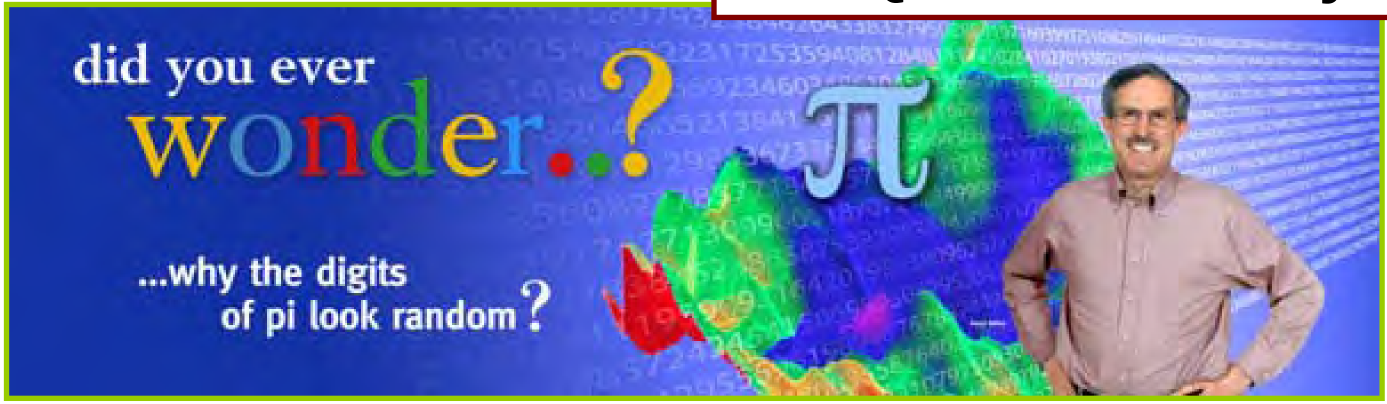
PiHex was a distributed computing project which used idle computing power to set three records for calculating specific bits of Pi. PiHex has now finished.

Position	Hex Digits Beginning At This Position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2
10^{11}	9C381872D27596
1.25×10^{12}	07E45733CC790B
2.5×10^{14}	E6216B069CB6C1

Undergraduate **Colin Percival's** grid computation **PiHex** rivaled **Finding Nemo**

1999 on 1736 PCS in 56 countries using 1.2 million Pentium2 cpu-hours

PSLQ and Normality of Digits



Bailey and Crandall observed that BBP numbers most probably are normal and make it precise with a hypothesis on the behaviour of a dynamical system.

- For example Pi is normal in Hexadecimal if the iteration below, starting at zero, is uniformly distributed in [0,1]

$$x_n = \left\{16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right\}$$

Consider the hex digit stream:

$$d_n = \lfloor 16x_n \rfloor$$

We have checked this gives first million hex-digits of Pi

Is this always the case? The weak Law of Large Numbers implies this is **very probably true!**

Pi to 1.5 trillion places in 20 steps

This fourth order algorithm was used on all big-Pi computations from 1986 to 2001

$$y_1 = \frac{1-\sqrt[4]{1-y_0{}^4}}{1+\sqrt[4]{1-y_0{}^4}}, a_1 = a_0\left(1+y_1\right)^4 - 2^3 y_1\left(1+y_1+y_1{}^2\right) \qquad y_{11} = \frac{1-\sqrt[4]{1-y_{10}{}^4}}{1+\sqrt[4]{1-y_{10}{}^4}}, a_{11} = a_{10}\left(1+y_{11}\right)^4 - 2^{23} y_{11}\left(1+y_{11}+y_{11}{}^2\right)$$

$$y_2 = \frac{1-\sqrt[4]{1-y_1{}^4}}{1+\sqrt[4]{1-y_1{}^4}}, a_2 = a_1\left(1+y_2\right)^4 - 2^5 y_2\left(1+y_2+y_2{}^2\right) \qquad y_{12} = \frac{1-\sqrt[4]{1-y_{11}{}^4}}{1+\sqrt[4]{1-y_{11}{}^4}}, a_{12} = a_{11}\left(1+y_{12}\right)^4 - 2^{25} y_{12}\left(1+y_{12}+y_{12}{}^2\right)$$

$$y_3 = \frac{1-\sqrt[4]{1-y_2{}^4}}{1+\sqrt[4]{1-y_2{}^4}}, a_3 = a_2\left(1+y_3\right)^4 - 2^7 y_3\left(1+y_3+y_3{}^2\right) \qquad y_{13} = \frac{1-\sqrt[4]{1-y_{12}{}^4}}{1+\sqrt[4]{1-y_{12}{}^4}}, a_{13} = a_{12}\left(1+y_{13}\right)^4 - 2^{27} y_{13}\left(1+y_{13}+y_{13}{}^2\right)$$

$$y_4 = \frac{1-\sqrt[4]{1-y_3{}^4}}{1+\sqrt[4]{1-y_3{}^4}}, a_4 = a_3\left(1+y_4\right)^4 - 2^9 y_4\left(1+y_4+y_4{}^2\right) \qquad y_{14} = \frac{1-\sqrt[4]{1-y_{13}{}^4}}{}$$

$$y_5 = \frac{1-\sqrt[4]{1-y_4{}^4}}{1+\sqrt[4]{1}}$$

$$- 2^{33} y_{16}\left(1+y_{16}+y_{16}{}^2\right)$$

These equations specify an algebraic number: $1/\pi$ □ a_{20}

$$y_7 = \frac{1-\sqrt[4]{1-y_6{}^4}}{1+\sqrt[4]{1-y_6{}^4}}, a_7 = a_6\left(1+y_7\right)^4 - 2^{15} y_7\left(1+y_7+y_7{}^2\right) \qquad y_{17} = \frac{1-\sqrt[4]{1-y_{16}{}^4}}{1+\sqrt[4]{1-y_{16}{}^4}}, a_{17} = a_{16}\left(1+y_{17}\right)^4 - 2^{35} y_{17}\left(1+y_{17}+y_{17}{}^2\right)$$

$$y_8 = \frac{1-\sqrt[4]{1-y_7{}^4}}{1+\sqrt[4]{1-y_7{}^4}}, a_8 = a_7\left(1+y_8\right)^4 - 2^{17} y_8\left(1+y_8+y_8{}^2\right) \qquad y_{18} = \frac{1-\sqrt[4]{1-y_{17}{}^4}}{1+\sqrt[4]{1-y_{17}{}^4}}, a_{18} = a_{17}\left(1+y_{18}\right)^4 - 2^{37} y_{18}\left(1+y_{18}+y_{18}{}^2\right)$$

$$y_9 = \frac{1-\sqrt[4]{1-y_8{}^4}}{1+\sqrt[4]{1-y_8{}^4}}, a_9 = a_8\left(1+y_9\right)^4 - 2^{19} y_9\left(1+y_9+y_9{}^2\right) \qquad y_{19} = \frac{1-\sqrt[4]{1-y_{18}{}^4}}{1+\sqrt[4]{1-y_{18}{}^4}}, a_{19} = a_{18}\left(1+y_{19}\right)^4 - 2^{39} y_{19}\left(1+y_{19}+y_{19}{}^2\right)$$

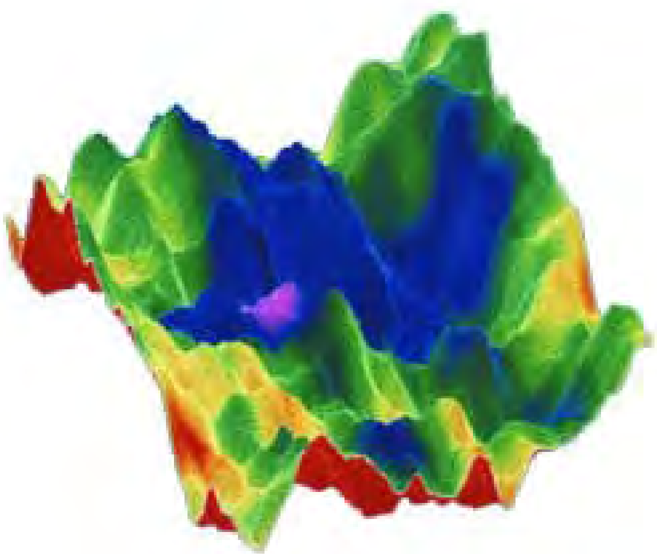
$$y_{10} = \frac{1-\sqrt[4]{1-y_9{}^4}}{1+\sqrt[4]{1-y_9{}^4}}, a_{10} = a_9\left(1+y_{10}\right)^4 - 2^{21} y_{10}\left(1+y_{10}+y_{10}{}^2\right) \qquad y_{20} = \frac{1-\sqrt[4]{1-y_{19}{}^4}}{1+\sqrt[4]{1-y_{19}{}^4}}, \mathbf{a_{20}} = a_{19}\left(1+y_{20}\right)^4 - 2^{41} y_{20}\left(1+y_{20}+y_{20}{}^2\right).$$

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

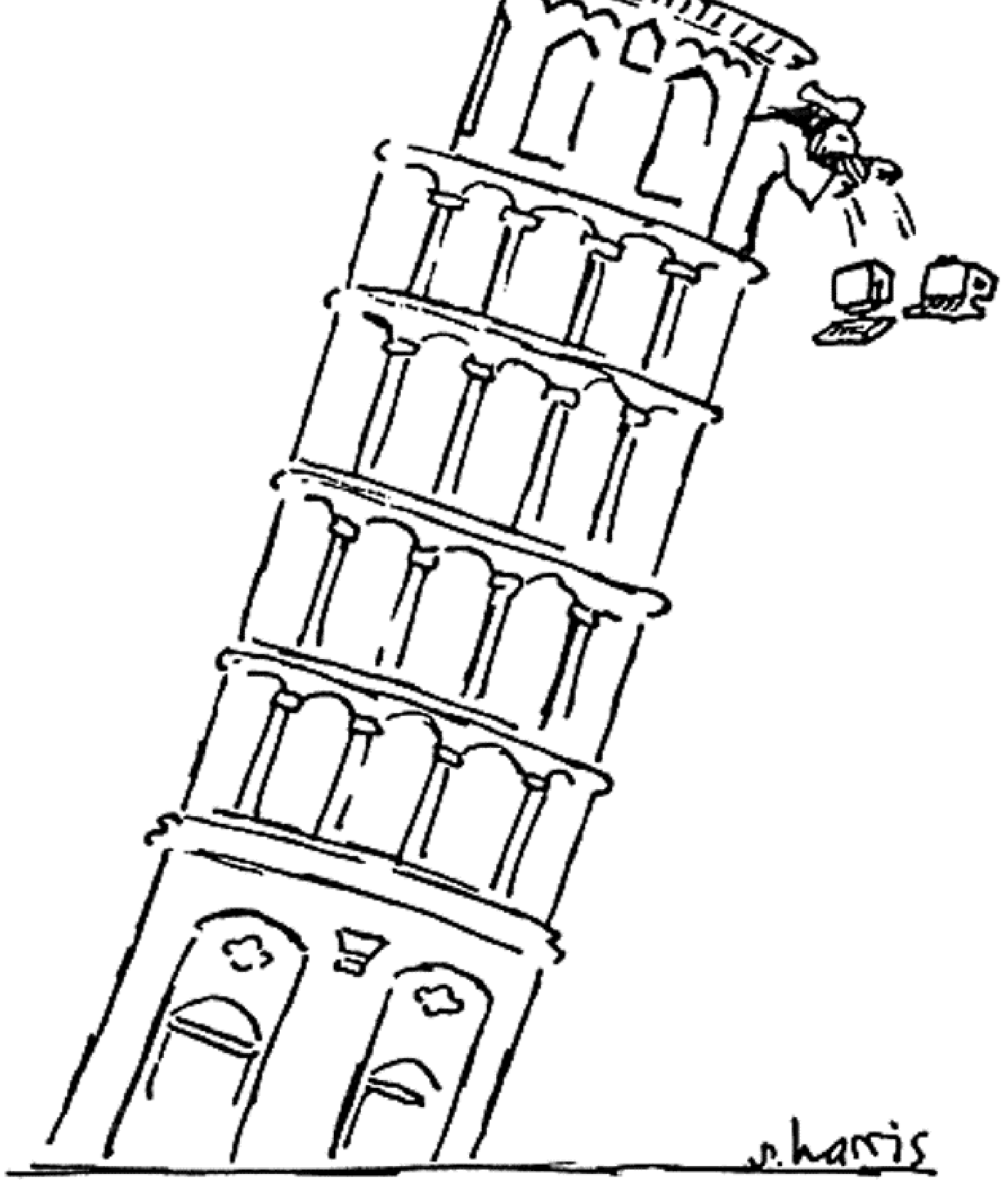
$$y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \quad \text{and}$$

$$a_{k+1} = a_k(1+y_{k+1})^4 - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2).$$

Then $1/a_k$ **converges quartically** to π



A random walk on a million digits of Pi



IF THERE WERE COMPUTERS IN GALILEO'S TIME

Knuth's Problem

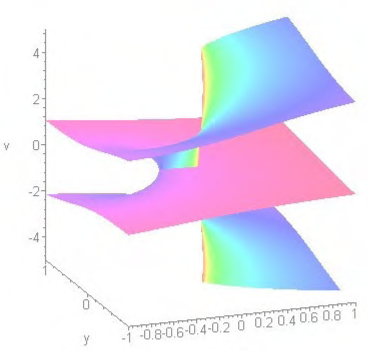
Donald Knuth* asked for a closed form evaluation of:

$$\sum_{k=1}^{\infty}\left\{\frac{k^k}{k!\,e^k}-\frac{1}{\sqrt{2\pi k}}\right\} = -0.084069508727655\ldots.$$

A guided proof followed on **asking why** Maple could compute the answer so fast.

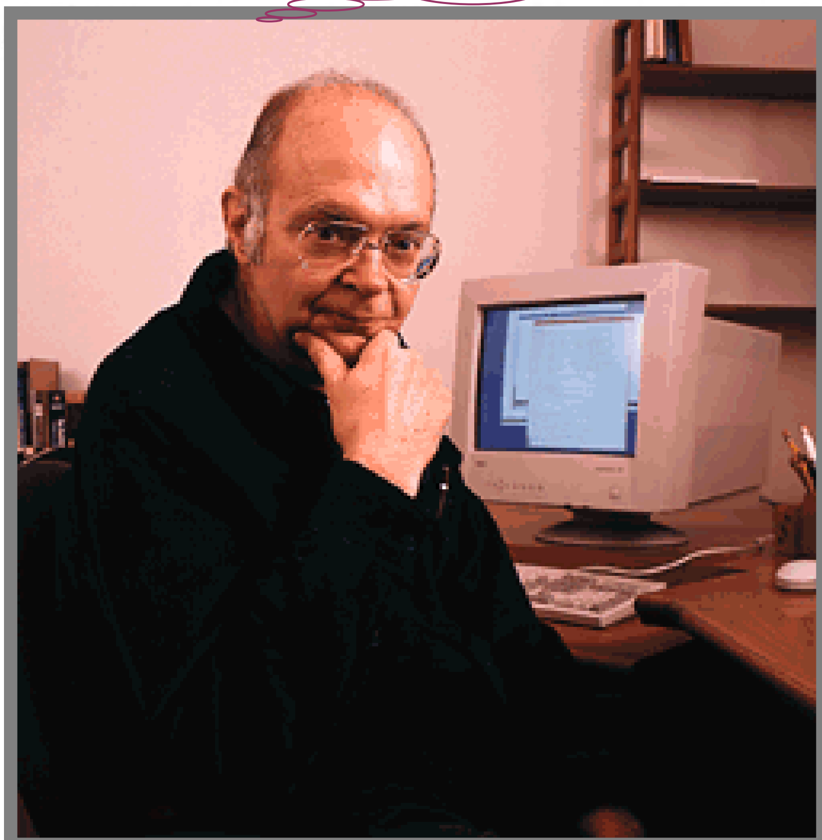
The answer is **Lambert's W** which solves

W exp(W) = x



W's **Riemann** surface

"instrumentation"



te 20 or 200 digits

shown on next slide

in the *Inverse Sym-*

turns

$$\approx \frac{2}{3}+\frac{\zeta(1/2)}{\sqrt{2\pi}}.$$

ich *Maple* 9.5 on a

in under 6 seconds

*** ARGUABLY WE ARE DONE**

ENTERING

evalf(Sum(k^k/k!/exp(k)-1/sqrt(2*Pi*k),k=1..infinity),16)

'Simple Lookup' fails;
'Smart Look up' gives:

INVERSE SYMBOLIC CALCULATOR

The ISC is the **Inverse Symbolic Calculator**, a set of programs and specialized tables of mathematical constants dedicated to the identification of real numbers. It also serves as a way to produce identities with functions and real numbers. It is one of the main ongoing projects at the Centre for Experimental and Constructive Mathematics (CECM).

Results of the search:

Maple output:

.08406950872765600

.84069508727656000e-1

Value to be looked up: .84069508727656000e-1 = K

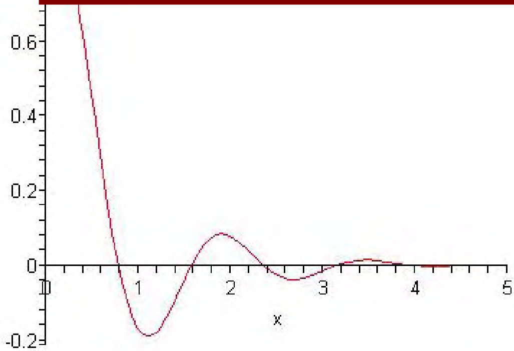
```
Performing a smart lookup on .84069508727656000e-1:
```

Function	Result	Precision	Matches
K-2/3	.5825971579390106666666666	16	1

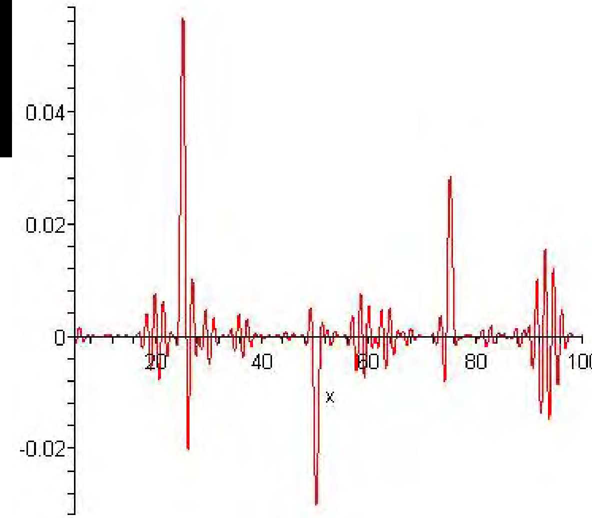
```
579390106 was probably generated by one
s or found in one of the given tables.
Answers are given from shortest to longest description

Mixed constants with 5 operations
5825971579390106 = Zeta(1/2)/sr(2)/sr(Pi)
```

Browse around .5825971579390106.

Quadrature I. Pi/8?



A numerically challenging integral



$$\int_0^\infty \cos(2x) \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right)\, dx \stackrel{?}{=} \frac{\pi}{8}.$$

But **Pi/8** equals

0.392699081698724154807830422909937860524645434

while the **integral** is

0.392699081698724154807830422909937860524646174

A careful *tanh-sinh quadrature* **proves** this difference after **43 correct digits**

Fourier analysis explains this happens when a hyperplane meets a hypercube

Before and After

Quadrature II. Hyperbolic Knots

$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt \stackrel{?}{=} L_{-7}(2) \qquad (@)$$

where

$$L_{-7}(s) = \sum_{n=0}^{\infty} \left[\frac{1}{(7n+1)^s} + \frac{1}{(7n+2)^s} - \frac{1}{(7n+3)^s} + \frac{1}{(7n+4)^s} - \frac{1}{(7n+5)^s} - \frac{1}{(7n+6)^s} \right].$$

"Identity" (@) has been verified to 20,000 places. I have *no idea* of how to prove it.

The easiest of 998 empirical results linking physics/topology (LHS) to number theory (RHS). [JMB-Broadhurst]

We have certain knowledge without proof

Extreme Quadrature … 20,000 Digits (50 Certified) on 1024 CPUs

- Ш. The integral was split at the nasty interior singularity
- Ш. The sum was `easy'.
- Ш. All fast arithmetic & function evaluation ideas used

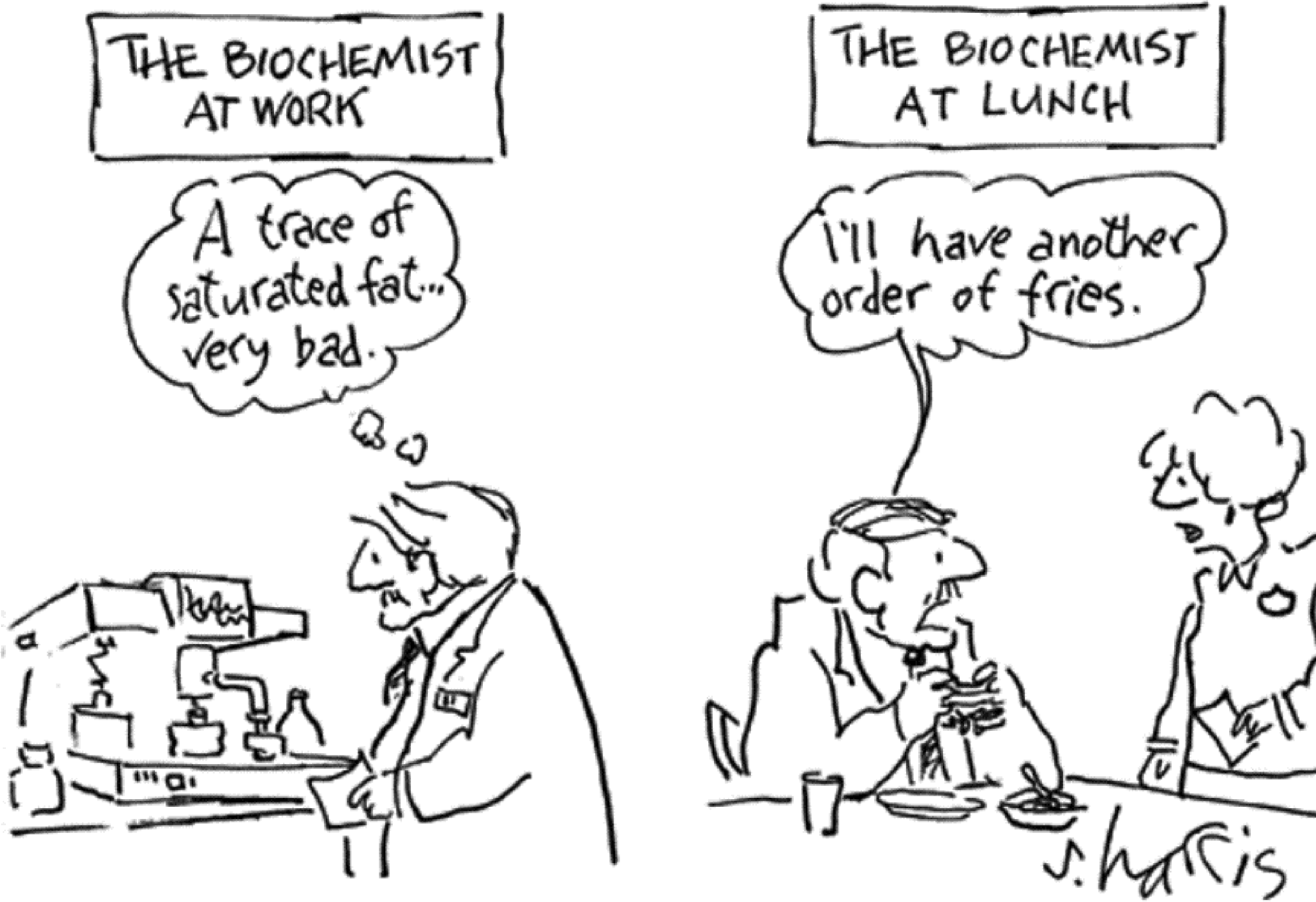
Run-times and speedup ratios on the Virginia Tech G5 Cluster

CPUs	Init	Integral #1	Integral #2	Total	Speedup
1	*190013	*1534652	*1026692	*2751357	1.00
16	12266	101647	64720	178633	15.40
64	3022	24771	16586	44379	62.00
256	770	6333	4194	11297	243.55
1024	199	1536	1034	2769	993.63

Parallel run times (in seconds) and speedup ratios for the 20,000-digit problem

Expected and unexpected scientific spinoffs

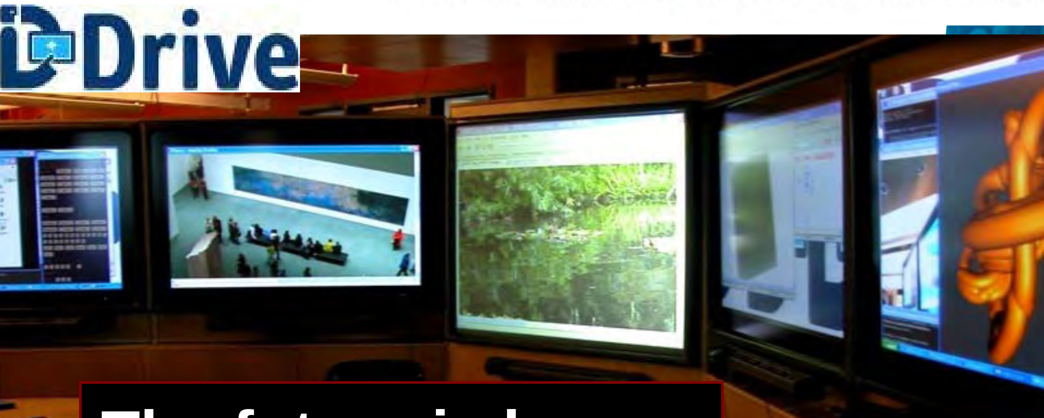
- **1986-1996**. Cray used quartic-Pi to check machines in factory
- **1986**. Complex FFT sped up by factor of two
- **2002**. Kanada used hex-pi (20hrs not 300hrs to check computation)
- **2005**. Virginia Tech (this integral pushed the limits)
- **1995-** Math Resources (another lecture)

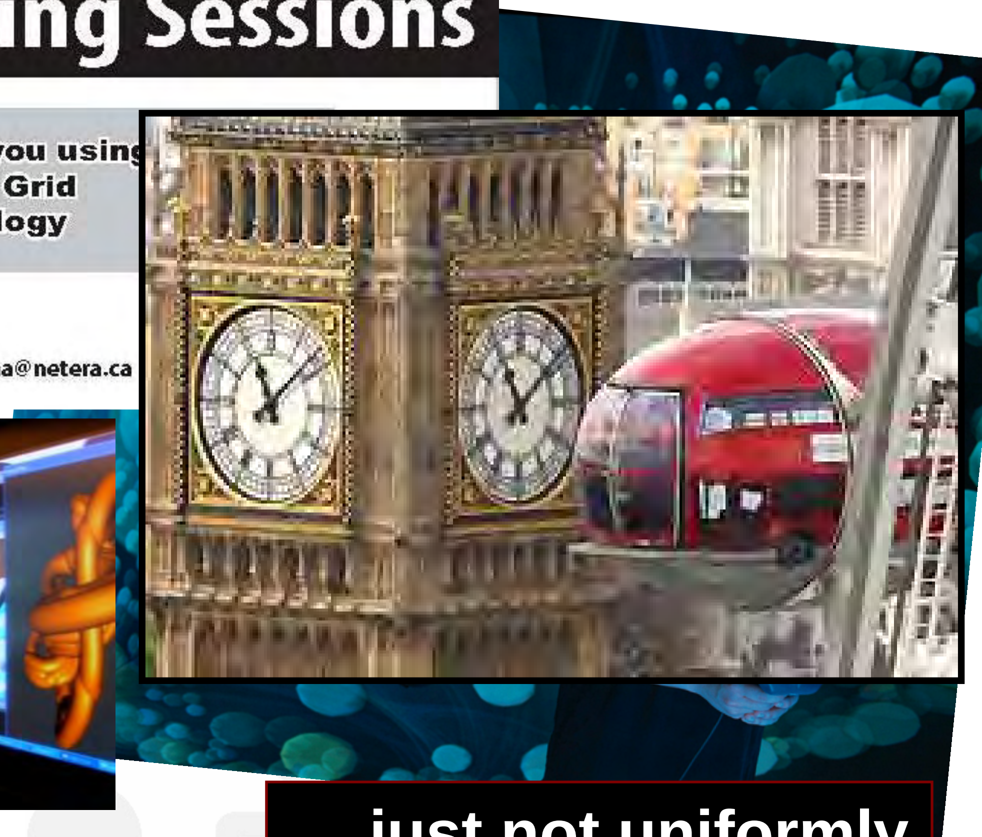

THE BIOCHEMIST AT WORK
A trace of saturated fat... very bad.
THE BIOCHEMIST AT LUNCH
I'll have another order of fries.
S. Harris

Chaitin Revisited

- Calude and Jurgenson (AAM 2005) prove a rigorous version of the following striking result that **there are lots more true results than provable ones**.

In this paper we prove Chaitin's "heuristic principle," *the theorems of a finitely-specified theory cannot be significantly more complex than the theory itself*, for an appropriate measure of complexity. We show that the measure is invariant under the change of the Gödel numbering. For this measure, the theorems of a finitely-specified, sound, consistent theory strong enough to formalize arithmetic which is arithmetically sound (like Zermelo–Fraenkel set theory with choice or Peano Arithmetic) have bounded complexity, hence every sentence of the theory which is significantly more complex than the theory is unprovable. Previous results showing that incompleteness is not accidental, but ubiquitous are here reinforced in probabilistic terms: the probability that a true sentence of length n is provable in the theory tends to zero when n tends to infinity, while the probability that a sentence of length n is true is strictly positive.



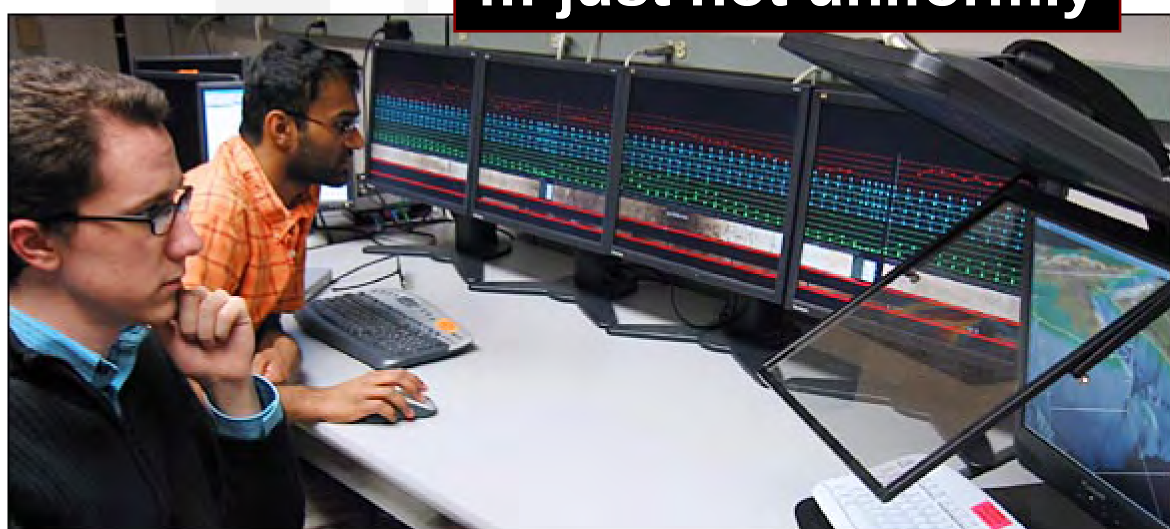


The future is here…

… just not uniformly

Remote Visualization via **Access Grid**

- The touch sensitive interactive **D-DRIVE**
- Immersion & haptics
- and the 3D **GeoWall**




THEORIES
PROOFS
S. Harris

REFERENCES

Enigma

J.M ... *ics by Experiment: Pla ...* A.K. Peters, 2003.

J.M ... sohn, *Ex ... tational Paths to Dis ...* **Active CDs 2006]**

D.H ... ntal Mathematics: Ex ... *otices Amer. Math. So ...*

"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."

- J. Hadamard quoted at length in E. Borel, *Lecons sur la theorie des fonctions*, 1928.