Exploring strange functions
on the computer
Continuous, nowhere differentiable functions

Weierstraß, 1872:

\[ C_{a,b}(x) := \sum_{n=0}^{\infty} a^n \cos(b^n \cdot b\pi x) \quad (|a| < 1, \ b > 1) \]

is cnd if \( b \in 2\mathbb{N} + 1, \ ab > 1 + \frac{3}{2}\pi. \)

Hardy, 1916: \( C_{a,b}, S_{a,b} \) cnd if \( b \in \mathbb{R}, \ b > 1, \ ab \geq 1. \)

Simpler proof? (Freud, Kahane, Hata, Baouche/Dubuc, . . . )

Consider only \( b \in \mathbb{N}, \) in fact \( b = 2, \) and

\[ S_{a,2}(x) = \sum_{n=0}^{\infty} a^n \sin(2^{n+1}\pi x) \quad (|a| < 1) \]

on \([0,1].\)
Functional equations

For $S = S_{a,2}$:

$$S\left(\frac{x}{2}\right) = aS(x) + \sin(\pi x),$$

$$S\left(\frac{x+1}{2}\right) = aS(x) - \sin(\pi x).$$

In general: System (F) consisting of

$$f\left(\frac{x}{2}\right) = a_0f(x) + g_0(x) \quad \text{(F}_0\text{)}$$

$$f\left(\frac{x+1}{2}\right) = a_1f(x) + g_1(x) \quad \text{(F}_1\text{)}$$

on $[0,1]$, for given $|a_0|,|a_1| < 1$, $g_0, g_1 : [0,1] \to \mathbb{R}$ and unknown $f : [0,1] \to \mathbb{R}$.

Examples: 1) $S_{a,2}$ with $a_0 = a_1 = a$ and $g_0(x) = -g_1(x) = \sin(\pi x)$.
2) $C_{a,2}$ with $a_0 = a_1 = a$ and $g_0(x) = -g_1(x) = \cos(\pi x)$.
3) $T_a(x) := \sum_{n=0}^{\infty} a^n d(2^n x)$, $d(x) = \text{dist}(x, \mathbb{Z})$,

with $a_0 = a_1 = a$ and $g_0(x) = \frac{x}{2}$, $g_1(x) = \frac{1-x}{2}$. 
Unique solutions?

\[ f \left( \frac{x}{2} \right) = a_0 f(x) + g_0(x), \quad f \left( \frac{x + 1}{2} \right) = a_1 f(x) + g_1(x). \quad (F) \]

\[ f \text{ solves } (F) \implies f(0) = \frac{g_0(0)}{1 - a_0}, \quad f(1) = \frac{g_1(1)}{1 - a_1} \]

\[ \implies f \left( \frac{1}{2} \right) = a_0 f(1) + g_0(1) = a_1 f(0) + g_1(0). \]

Thus: If a solution exists, then necessarily

\[ a_0 \frac{g_1(1)}{1 - a_1} + g_0(1) = a_1 \frac{g_0(0)}{1 - a_0} + g_1(0). \quad (*) \]

Moreover,

\[ f \left( \frac{1}{4} \right) = a_0 f \left( \frac{1}{2} \right) + g_0 \left( \frac{1}{2} \right), \quad f \left( \frac{3}{4} \right) = a_1 f \left( \frac{1}{2} \right) + g_1 \left( \frac{1}{2} \right), \]

\[ f \left( \frac{1}{8} \right) = a_0 f \left( \frac{1}{4} \right) + g_0 \left( \frac{1}{4} \right), \quad f \left( \frac{3}{8} \right) = \ldots, \quad f \left( \frac{5}{8} \right) = \ldots, \quad f \left( \frac{7}{8} \right) = a_1 f \left( \frac{3}{4} \right) + g_1 \left( \frac{3}{4} \right), \]

\[ f \left( \frac{2i+1}{16} \right), \]

\[ \ldots \]

\[ f \left( \frac{2i+1}{2^n} \right). \]
Schauder basis

\[ \sigma_{i,j}(x) \]

\[ \frac{1}{2^{j+1}} \quad \frac{2i+1}{2^j} \quad \frac{i+1}{2^j-1} \]

\[ \sigma_{0,1} \]

\[ \sigma_{0,2} \quad \sigma_{1,3} \quad \sigma_{2,3} \quad \sigma_{3,3} \]

\[ 0 \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad 1 \]
Schauder coefficients

Theorem (Schauder, 1930, and Faber, 1908).
Every \( f \in C[0, 1] \) has a unique expansion of the form

\[
f(x) = \gamma_{0,0}(f) \sigma_{0,0}(x) + \gamma_{1,0}(f) \sigma_{1,0}(x) + \sum_{n=1}^{\infty} \sum_{i=0}^{2^{n-1}-1} \gamma_{i,n}(f) \sigma_{i,n}(x),
\]

where the coefficients \( \gamma_{i,n}(f) \) are given by

\[
\gamma_{0,0}(f) = f(0), \quad \gamma_{1,0}(f) = f(1), \quad \text{and}
\]

\[
\gamma_{i,n}(f) = f \left( \frac{2i + 1}{2^n} \right) - \frac{1}{2} f \left( \frac{i}{2^{n-1}} \right) - \frac{1}{2} f \left( \frac{i + 1}{2^{n-1}} \right).
\]

Theorem (Faber, 1910).
Assume that \( f \in C[0, 1] \) has a finite derivative at some point \( x_0 \). Then

\[
\lim_{n \to \infty} 2^n \cdot \min_{n \to \infty} \{|\gamma_{i,n}(f)| : i = 0, \ldots, 2^{n-1} - 1\} = 0.
\]
Recursion formula for solutions of (F)

Theorem.

Assume that (*) holds and that \(g_0, g_1\) are continuous.

Let \(f\) be the continuous solution of the system (F).

Then

(i) \(\gamma_{0,0}(f) = f(0) = \frac{g_0(0)}{1-a_0}\) and \(\gamma_{1,0}(f) = f(1) = \frac{g_1(1)}{1-a_1}\),

(ii) \(\gamma_{0,1}(f) = \left(a_1 - \frac{1}{2}\right)f(0) - \frac{1}{2}f(1) + g_1(0) = \left(a_0 - \frac{1}{2}\right)f(1) - \frac{1}{2}f(0) + g_0(1),\)

(iii) \(\gamma_{i,n+1}(f) = a_0\gamma_{i,n}(f) + \gamma_{i,n}(g_0)\) for \(i = 0, \ldots, 2^{n-1} - 1,\)

\(\gamma_{i,n+1}(f) = a_1\gamma_{i-2^{n-1},n}(f) + \gamma_{i-2^{n-1},n}(g_1)\) for \(i = 2^{n-1}, \ldots, 2^n - 1.\)
Results and questions

Let \( \delta_n(f) := 2^n \cdot \min \{|\gamma_{i,n}(f)| : i = 0, \ldots, 2^{n-1} - 1\} \).

**Theorem.** \( \delta_n(S_{a,2}) \not\to 0 \ (n \to \infty) \) for \( 1 > a \geq \frac{1}{2} \).

This proves that \( S_{a,2} \) is cnd for \( 1 > a \geq \frac{1}{2} \).

**Open questions.**

1) Show that, for \( a = \frac{1}{2} \), \( \lim_{n \to \infty} \delta_n(S_{a,2}) \) exists, and find its value.

2) Show, more generally, that \( \lim_{n \to \infty} \delta_n(S_{a,2})/(2|a|)^n \) exists, and determine the function \( a \mapsto \lim_{n \to \infty} \delta_n(S_{a,2})/(2|a|)^n \).
A functional equation with discontinuous solution

Consider the system, for given $0 < q < 1$,

$$s\left(\frac{x}{2}\right) = q s(x) - 1,$$

$$s\left(\frac{x+1}{2}\right) = q s(x) + 1.$$ 

This system has a unique bounded solution $s_q$, which is discontinuous precisely at the dyadic rationals.

Let $F_q(t) := m\{x \in [0, 1] \mid s_q(x) \leq t\}$, the distribution function of $s_q$.

It can be shown that $F_q$ is the unique function satisfying the functional equation

$$F(t) = \frac{1}{2} F\left(\frac{t-1}{q}\right) + \frac{1}{2} F\left(\frac{t+1}{q}\right)$$

with $F_q(t) = 0$ for $t < -1/(1-q)$ and $F_q(t) = 1$ for $t > 1/(1-q)$.

**Theorem (Jessen/Wintner 1935).**

$F_q$ is either absolutely continuous or singular.

**Question:** For which $q$ is $F_q$ absolutely continuous, for which $q$ is it singular?
Some answers

Theorem (Kershner/Wintner 1935).
For $0 < q < \frac{1}{2}$, $F_q$ is singular (in fact, a Cantor function).

Theorem (Wintner 1935).
For $q = \frac{1}{2}$, $F_q(t) = \begin{cases} 0, & t < -2 \\ \frac{t + 2}{4}, & -2 \leq t \leq 2 \\ 1, & t > 2 \end{cases}$, which is absolutely continuous.
In fact, for each $q = 2^{-1/p}$, $F_q$ is absolutely continuous.

Theorem (Erdős 1939).
If $q > \frac{1}{2}$ and $1/q$ is a Pisot number, then $F_q$ is singular!
E.g., $F_q$ is singular for $q = (\sqrt{5} - 1)/2 \approx 0.618033989$.
(Proof: via the Fourier-Stieltjes transform of $F_q$.)

Theorem (Garsia 1962).
Some explicit algebraic numbers $q$ (besides $2^{-1/p}$) for which $F_q$ is absolutely continuous.

Theorem (Solomyak 1995).
$F_q$ is absolutely continuous for a.e. $q \in (\frac{1}{2}, 1)!$
Open questions and experimental approach

Open: 1) Is the set of exceptional values $q > \frac{1}{2}$ (with $F_q$ singular) countable?
2) Is there a rational $q > \frac{1}{2}$ with $F_q$ singular?
   Is there a rational $q > \frac{1}{2}$ with $F_q$ absolutely continuous?
3) What about $q = \frac{2}{3}$? What about other specific values?

Experimental approach: Visualize the density $f_q = F'_q$ a.e.
In fact, if $F_q$ is absolutely continuous, then $f_q$ is a non-trivial $L_1$-solution of the functional equation

$$f(t) = \frac{1}{2q} \left( f \left( \frac{t - 1}{q} \right) + f \left( \frac{t + 1}{q} \right) \right),$$

on $\mathbb{R}$.
Vice versa, if a non-trivial $L_1$-solution $f_q$ of $(S_q)$ exists, then it is the density of an absolutely continuous $F_q$. 
How to visualize $f_q$?

$$f(t) = \frac{1}{2q} \left( f \left( \frac{t - 1}{q} \right) + f \left( \frac{t + 1}{q} \right) \right)$$  \hspace{1cm} (S_q)

It can be shown: If a non-trivial $L_1$-solution $f_q$ of $(S_q)$ exists, then it:

- is unique up to a multiplicative constant,
- satisfies $\text{supp } f_q = \left[-\frac{1}{1-q}, \frac{1}{1-q}\right]$,
- and is either positive or negative a.e. on its support.

This implies: Define an operator $B_q$ on $L_1$ by

$$(B_q f)(t) = \frac{1}{2q} \left( f \left( \frac{t - 1}{q} \right) + f \left( \frac{t + 1}{q} \right) \right)$$

and consider the iteration $f^{(n)} := B_q f^{(n-1)}$ with some $f^{(0)} \in L_1$. Then:

If $(f^{(n)})_n$ converges in $L_1$, then the limit is an $L_1$-solution of $(S_q)$.

If $(S_q)$ has a non-trivial $L_1$-solution, then $(f^{(n)})_n$ converges in the mean in $L_1$. 

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A final remark about $q = 2/3$

Rescale $F_q$ resp. $f_q$ such that the support is $[0, 1]$ instead of $[-\frac{1}{1-q}, \frac{1}{1-q}]$.

Then for $q = 2/3$, the functional equation $(S_q)$ is equivalent to the system

$$f \left( \frac{x}{3} \right) = \frac{3}{4} f \left( \frac{x}{2} \right),$$

$$f \left( \frac{x+1}{3} \right) = \frac{3}{4} f \left( \frac{x}{2} \right) + \frac{3}{4} f \left( \frac{x+1}{2} \right),$$

$$f \left( \frac{x+2}{3} \right) = \frac{3}{4} f \left( \frac{x+1}{2} \right)$$

on $[0, 1]$.

Does this system have a non-trivial $L_1$-solution?

If so, is the solution continuous?