The evaluation of definite integrals

Experimental Mathematics Workshop

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Victor H. Moll
Department of Mathematics
Tulane University
New Orleans

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Central Problem

Given a function f and a, $b \in \mathbb{R} \cup \{\pm\infty\}$

evaluate

$$I(f; a, b) := \int_a^b f$$

in terms of the parameters of f and a, b

What is an answer?

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{6}$$

$$\int_0^{1/5} \frac{dx}{\sqrt{1-x^2}} = \mathrm{Arcsin}(1/5)$$

Values of special functions

$$\int_{-1}^{1} \frac{e^{2x}\,dx}{\sqrt{1-x^2}} = \pi\, I_0(2)$$

where $I_0(z)$ is the Bessel function

of imaginary argument:

$$I_0(z) = J_0(\sqrt{-1}z) = \sum_{k=0}^{\infty} \frac{1}{k!^2} \left(\frac{z}{2}\right)^{2k}$$

Indefinite integrals

Complete theory: Abel, Liouville, Risch, ...

$$\int \ln(1+x)\,dx = (1+x)\,\ln(1+x) - x$$

$$f_0(x) = \ln(1+x) \qquad f_n(x) = \int f_{n-1}(x)\,dx$$

$$f_1(x) = \frac{(x+1)^2}{2}\,\ln(1+x) - \frac{x(3x+2)}{4}$$

$$f_n(x) = -xC_n(x) + B_n(x)\,\ln(1+x)$$

$$B_n(x) = \frac{(1+x)^n}{n!}$$

Interesting parts

$$C_n(x) = \frac{D_n(x)}{a_n} \qquad D_n \in \mathbb{Z}[x]$$

$$a_n = n! \times 10^{\sum_{k=0}^{n} \Lambda(k)}$$

$$\Lambda(n) = \begin{cases} \log p \ \text{ if } \ n = p^k \,\text{for}\, p \,\text{prime} \\ 0 \text{ otherwise} \end{cases}$$

von Mangoldt's function

$$D_n(x) = ???????$$

There is no theory for definite integrals

Complexity level is hard to predict

$$\int_{-\infty}^{\infty} \frac{dx}{(e^x - x + 1)^2 + \pi^2} = \frac{1}{2}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(e^x - x)^2 + \pi^2} = \frac{1}{1 - W(1)}$$

$W(z)$ is the Lambert function

$$W(z) \ \exp W(z) = z$$

Constants

$$\int_0^\infty \ln\left(\frac{ax+b}{bx+a}\right)\frac{dx}{(1+x)^2} = 0$$

Special values of elementary functions

$$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^{2x}-1}} = 1 - \ln 2$$

Radicals

$$\int_0^\infty \frac{dx}{(x^8+5x^6+2x^4+5x^2+1)^3} = \frac{3\pi(647626+172309\sqrt{14})}{43904\sqrt{608409+325204\sqrt{14}}}$$

Other constants

Euler's constant:

$$\gamma := \lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

$$\int_0^1 \left(x - \frac{1}{1 - \ln x} \right) \frac{dx}{x \, \ln x} = \gamma$$

$$\text{Catalan: } G := \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2}$$

$$\int_0^\infty \frac{x \, dx}{\cosh x} = 2G$$

Vardi's evaluation

$$\int_{\pi/4}^{\pi/2} \ln \ln \tan x \, dx = \frac{\pi}{2} \ln \left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \cdot \sqrt{2\pi} \right)$$

uses Dirichlet's L-functions

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots$$

An interesting new addition

Gradshteyn-Rhyzik 3.248.5:

$$\varphi(x) \;=\; 1 + \frac{4x^2}{3(1+x^2)}$$

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[\varphi(x) + \sqrt{\varphi(x)}\right]^{1/2}} \;=\; \frac{\pi}{2\sqrt{6}}$$

Unfortunately it is wrong.

What are these good for?

$$\int_0^1 \frac{(1-x)^4\, x^4}{1+x^2}\, dx \;=\; \frac{22}{7} - \pi$$

so that $\pi \neq \frac{22}{7}$.

Problem: Find a (nice) rational function R such that

$$\int_0^1 R(x)\, dx \;=\; \pi - \frac{355}{113}$$

Sometimes we get the same answer

$$\int_0^1 \frac{\ln x \, dx}{\sqrt[3]{x(1-x^2)^2}}$$

$$\int_0^1 \frac{\ln(x+1/x) \, dx}{(1+x^2)\sqrt[3]{x(1+x^2)^2}}$$

have the same value

$$\left[\frac{1}{2}\Gamma\left(\frac{1}{3}\right)\right]^3$$

Kontsevich-Zagier period conjecture

period: $\int_\Omega R(z)$
$R \in \mathbb{Q}(z)$
$\Omega \subset \mathbb{R}^n$
defined by polynomial equations over $\mathbb{Q}$.

Conjecture.

There is a (well-defined) class
of transformations that connect
both representations of a period.

1) Additivity, 2) Change of variables,
3) Perfect derivative.

Wallis' formula: $\sim$ 1655

$$\frac{1}{\pi}\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} \;=\; 2^{-2m-1}\binom{2m}{m}$$

What happens for higher degree?

A quartic integral

$$
\begin{aligned}
N_{0,4}(a;m) &:= \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \\
&= \frac{\pi}{2^{m+3/2}\,(a+1)^{m+1/2}} P_m(a)
\end{aligned}
$$

where

$$
P_m(a) = \sum_{l=0}^{m} d_l(m) a^l
$$

with

$$
d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}
$$

The polynomial Riemann hypothesis

The coefficients $d_l(m)$ satisfy

$$d_l(m) = \frac{1}{l!\, m!\, 2^{m+l}} \times$$

$$\left(\alpha_l(m) \prod_{k=1}^{m} (4k-1) - \beta_l(m) \prod_{k=1}^{m} (4k+1) \right)$$

where α_l and β_l are polynomials in m.

Theorem (John Little, 2003). The zeros of the families $\alpha_l(m)$ and $\beta_l(m)$ are on the line $Re(m) = -1/2$.

The double square root

Theorem: (Boros, G. - V.M., 2001)

$$\sqrt{a+\sqrt{1+x}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \times$$
$$\times \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N_{0,4}(a;k-1)x^k$$

Conjecture

$$\sqrt{a+\sqrt{b+\sqrt{1+x}}} = \sum_{n=0}^{\infty} \beta_n[a,b]x^n$$

$$\beta_n[a,b] = \frac{(-1)^{n-1}}{n2^{2n+1}} \sum_{k=0}^{n-1} \binom{2n-2-k}{n-1} \times$$
$$\times \frac{P_k^*(a,\sqrt{1+b})}{[(1+b)(a+\sqrt{1+b})]^{k+1/2}}$$

$$P_k^*(a,z) = z^k P_k(a/z)$$

LANDEN TRANSFORMATIONS

Rational case: degree 6
Boros,G. - V.M. 1997

$$U(a,b;c,d,e) \;=\; \int_0^\infty \frac{cx^4+dx^2+e}{x^6+ax^4+bx^2+1}\,dx$$

is invariant under the transformation

$$\begin{aligned} a_{n+1} &= T^{-4/3}\,(a_nb_n+5a_n+5b_n+9) \\ b_{n+1} &= T^{-2/3}\,(a_n+b_n+6) \end{aligned}$$

with $T = a_n + b_n + 6$ and similar formulas for c_n, d_n and e_n.

HIGHER DEGREE

Theorem. (Boros, G. - V.M., 2000)

Given an even rational function $R(x)$ there exist an explicit transformation of the coefficients

$$a_{n+1} \;=\; F(a_n, b_n, \cdots)$$

that preseves the integral of R.

CONVERGENCE

Theorem. (Hubbard, J. - V.M., 2001)

Assume that the initial parameters are such that the integral of R is finite. Then the previous algorithm converges quadratically to the 1-form $L\,dz/(z^2+1)$.

The original integral can now be evaluated by iteration.

Idea of proof: The map $w = \pi(z) = (z^2-1)/2z$ is conjugate to $F(z) = z^2$ via $M(z) = (z+i)/(z-i)$.

original integrand: $\varphi = R(z)\,dz$
new one:

$$\pi_* R(z)\,dz := R(\sigma_+(w))\frac{d\sigma_+}{dw} + R(\sigma_-(w))\frac{d\sigma_-}{dw}$$

with

$$\sigma_\pm(w) = w \pm \sqrt{w^2+1}$$

The Tornheim-Zagier sums

$$T(a,0,c) = 4\lambda(a)\lambda(c)\sin(\pi c/2)\times$$

$$\sin\left(\frac{\pi a}{2}\right)\left[\zeta(1-a)\zeta(1-c) - \frac{\zeta(1-a-c)B(a,c)}{1-g(a)g(c)}\right]$$

$$-\frac{1}{2}\cos\left(\frac{\pi a}{2}\right)\times\int_0^1 \zeta(1-c,q)\zeta_-(1-a,q)\cot(\pi q)\,dq$$

$$\lambda(z) = \Gamma(1-z)\,(2\pi)^{z-1}$$

$$g(z) = \tan\left(\frac{\pi z}{2}\right)$$

$$\zeta_-(1-a,q) = \zeta(1-a,q) - \zeta(1-a,1-q)$$

The Tornheim-Zagier sums

$$T(n,0,m) = L_1(n,m) + L_2(n,m)$$

$$L_1(n,m) = \frac{1}{4}\frac{(2\pi)^{n+m}}{(n-1)!(m-1)!}\times$$

$$(Z(n)Z(m) - B(n,m)Z(n+m))$$

$$Z(n) = \begin{cases} (-1)^{n/2}\zeta(1-n), & n \text{ even} \\ \\ \frac{2}{\pi}(-1)^{(n-1)/2}\zeta'(1-n), & n \text{ odd} \end{cases}$$

The Tornheim-Zagier sums

n and m even

$$L_2(n,m) = \frac{2(2\pi)^{n+m-2}}{n!\,m!}(-1)^{(n+m)/2}\times$$

$$(nW(n-1,m) + mW(n,m-1))$$

$$W(i,j) = \int_0^1 A_i(q)B_j(q)\ln\sin\pi q\ dq$$

$$T(2,0,6) = \frac{\pi^8}{8100} + \frac{8\pi^6}{45}\left(W(1,6) + 3W(2,5)\right)$$

Adamchik, 2004

$$\frac{A_k(q)}{k} =$$

$$\zeta'(1-k)+\sum_{j=0}^{k-1}(-1)^{k-1-j}j!Q_{j,k-1}(q)\ \log\Gamma_{j+1}(q)$$

$$Q_{k,n}(q) = \sum_{j=k}^{n}(1-q)^{n-j}\binom{n}{j}\left\{ {j \atop k} \right\}$$

$$\Gamma_{n+1}(q+1) = \frac{\Gamma_{n+1}(q)}{\Gamma_n(q)}$$

$$\Gamma_1(q) = \Gamma(q)$$

$$\Gamma_n(1) = 1$$

Reduction

$$T(a,b,c) = T(a,b{-}1,c{+}1){+}T(a{-}1,b,c{+}1)$$

$$T(a,b,c) = T(b,a,c)$$

$$T(a,b,0) = \zeta(a)\,\zeta(b)$$

$$T(a,b,c) = \sum_{i=1}^{a} \binom{a+b-i-1}{a-i} T(i,0,N{-}i){+}$$
$$\sum_{i=1}^{b} \binom{a+b-i-1}{b-i} T(i,0,N-i)$$

$$\text{where } N = a+b+c$$

For N odd at least 3:

$$T(i,0,N-i) =$$

$$(-1)^i \sum_{j=0}^{(N-i-1)/2} \binom{N-2j-1}{i-1} \zeta(2j)\zeta(N-2j) +$$

$$+(-1)^i \sum_{j=0}^{i/2} \binom{N-2j-1}{N-i-1} \zeta(2j)\zeta(N-2j)$$

$$+\zeta(0)\zeta(N)$$

Open question for N even: first case $T(2,0,6)$

T(2,0,6) requires

$$\int_0^1 B_5(q)\ \log\Gamma(q)\ \log\sin\pi q\ dq$$

$$\int_0^1 B_6(q)\ \log\Gamma(q)\ \log\sin\pi q\ dq$$

$$\int_0^1 B_5(q)\ \log\Gamma_2(q)\ \log\sin\pi q\ dq$$

$\Gamma_2(q) = 1/G(q),$ Barnes function