BOOK REVIEW PIONEERS UNDERSCORE REVOLUTION

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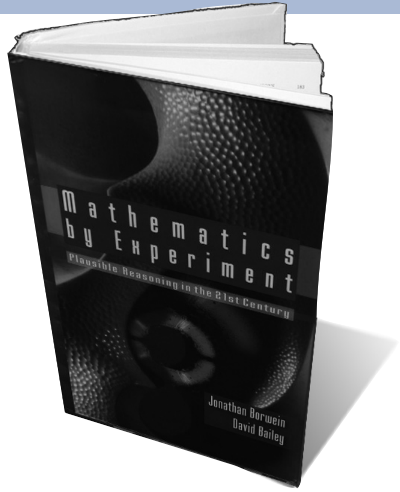
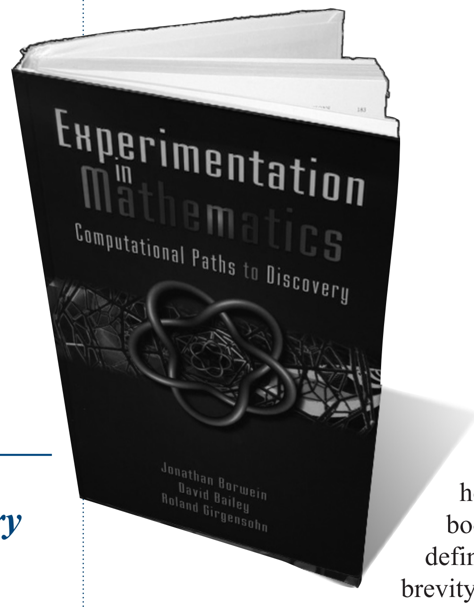




Mathematics by Experiment: Plausible Reasoning in the 21st Century
by Jonathan Borwein and David Bailey
A. K. Peters 2004, x + 288 pp.

Experimentation in Mathematics: Computational Paths to Discovery
by Jonathan Borwein, David Bailey, and Roland Girgensohn, A. K. Peters 2004, x + 357 pp.

This reviewer is compelled to admit right off: Perusal of these two books reminded me of my own college days at CalTech. You see, everything was fresh then, and I speak not merely of youth *per se*. Much of the scientific culture was driven in that era by the world view of the eminent physicist Richard Feynman. As brash as he was, he was also a pioneer of ideas—and not all of these ideas were immediately accepted by the professional community. I always thought that Feynman's life mission was not just to throw off the shackles of accepted science, but also to *replace* the past with something new; and from this reviewer's point of view, nothing is more intellectually exciting than that.

These two books, written by pioneers in the field of experimental mathematics, have provided us with a crisp snapshot of the state of that field at the dawn of this new century. Given that the general public embraced computers as "number-crunching" engines—and little more—for the first several decades of computing, it is really, now, in the early 2000s about time for computers to weld solidly to pure mathematics. These marvelous books underscore this revolution (there have been at least four computer revolutions by now; this reviewer is claiming experimental mathematics is yet another one), and such books absolutely have to be written, and read. This is the Feynmanesque excitement, then: Everything in this experimental field is fresh right now; moreover, the computational past is not merely being rejected, it is—at least in part—being replaced.

For readers who are not familiar with the historical and technical nuances of the phrase "experimental mathematics," a word here is appropriate. In the first book, the authors give a detailed definition which I paraphrase for brevity here:

"Experimental Mathematics *is that branch of mathematics that concerns itself ultimately with...insights...through the use of experimental...exploration...*"

The details omitted here give the authors' defining paragraph a philosophical bent; but for many readers their simpler, introductory definition

"*This new approach to mathematics—the utilization of advanced computing technology in mathematical research—is often called experimental mathematics*."

should suffice. Though it could be argued that experimental mathematicians have been around virtually forever (Gauss himself is certainly a candidate for the moniker), the modern era of experimental mathematics can be said to have begun in the early 1990s. The year 1992 saw the advent of the journal *Experimental Mathematics*, and one of the canonical anecdotes of the field was the 1993 *numerical* discovery by Enrico Au-Yeung—a student at U. Waterloo—that

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)^2 k^{-2} \approx \frac{17\pi^4}{360}$$

in the sense that equality holds here to at least six significant decimals. One reads that it came to the attention of Borwein, Bailey, and Girgensohn that Au-Yeung's relation even held to 30 and later to 100 decimals, and eventually the relation was rigorously proved. This development in the experimental arena is especially refreshing in that here in the 2000s, there are extra reasons to take numerical work into the realms of extreme precision, thus leaking bright light once and for all into the "number-crunching" limbo of history.

Beyond such sparkling anecdotes, the books lead us through a forest of these experimental conjectures, and in many cases proofs of same. There are polylogarithmic discoveries in connection with knot theory and particle physics, problems from probability and statistics, and much more. It should be stressed that many identities remain unproven, even though they hold to thousands of decimals.

The books go well beyond just identity-finding and subsequent proving. There are discussions of fast arithmetic *per se*, Fourier transforms, normal numbers and digit expansions generally, some number theory, entropy principles, and on and on. There is—as expected, given the historical backgrounds of the authors—a lot about the number π, including the celebrated Bailey–Borwein–Plouffe formula that has led computationalists to know that the quadrillionth binary bit of π is a zero. This "BBP" formula has also led to some theorems and expectations on how normality proofs might go, in future, for certain fundamental constants. One important component of the books is the discussion of PSLQ, worked out by sculptor-mathematician Helaman Ferguson, as a computational scheme for finding linear relations of the form

$$r_1 x_1 + \cdots + r_n x_n = 0$$

where the x_k are known each to say hundreds or thousands of digits, and the PSLQ system finds *integer* coefficients r_k, or rules out the existence of the same, within norm limits on said coefficients. For example, the left-hand side of Au-Yeung's formula above could be x_1, to say 300 digits, and x_2 could be π^4 (or $\zeta(4)$) also to high precision, and PSLQ will report the Au-Yeung identity itself. This is not the same as proving, of course, but this reviewer believes that by today, 2006, the PSLQ relation-detection system has inspired about as many proofs as a good career teacher can inspire in a lifetime. And that should not be disturbing, for PSLQ is after all a human invention.

This reviewer wishes to convey a more personal anecdote in connection with these fine books. While perusing these books occasionally over several months, a problem concerning the Riemann zeta function at extreme imaginary height occupied my thoughts, and at one point I was about to give up because a certain expansion seemed not to have effectively boundable coefficients. Then by leafing serendipitously once again through these books, I realized the Lambert *W*-function was the creature at hand. What is more, the theory and examples and problems posed by the authors for this *W* amounted to a prefabricated tour of what I had to get through to solve my own problem! This little experience all by itself made the books precious to me.

One supposes a believable review must mention some drawbacks or imperfections, yet there are only minor ones in these books. The index (of each volume) could be perhaps significantly more dense. But the only real complaint this reviewer has is that he wishes there were just one 600-page book, because I need to take the books around often enough, and it is awkward to travel with a two-piece "experimental bible."

In closing, back once more to Feynman and his world view:

"The worthwhile problems are the ones you can really solve or help solve, the ones you can really contribute something to."

-Richard Feynman, 1966

So, in these volumes, the brave pioneers Borwein, Bailey, and Girgensohn have not only chosen worthwhile problems, in Feynman's sense of accessibility, but also shown us the pathways that will allow many of us readers to select worthwhile problems in the sense of personal achievement. These are two beautiful books, both of which belonging squarely on the desk of any aspiring discoverer, or—might I say it?—on the shelf of any historian of science.

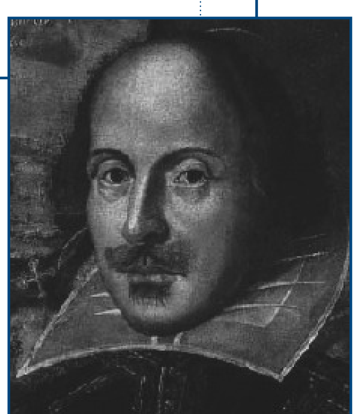
I do present you with a man of mine,

Cunning in music and in mathematics,

To instruct her fully in those sciences,

Whereof, I know, she is not ignorant.

William Shakespeare, Taming of the Shrew, Act 2, Sc.1.



Music and poesy used to quicken you:

The mathematics, and the metaphysics,

Fall to them as you find yur stomach serves you.

No profit grows, where is no pleasure ta'en:--

In brief, sir, study what you most affect.

William Shakespeare,
Taming of the Shrew, Act 1, Sc.1.