Some questions in the evaluation of definite integrals

MAA Short Course

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Central Problem

Given a function f and $a,\, b \in \mathbb{R} \cup \{\pm\infty\}$

evaluate

$$I(f; a, b) := \int_a^b f$$

in terms of the parameters of f and $a,\, b$

There is no theory for definite integrals

Complexity level is hard to predict

$$\int_{-\infty}^{\infty} \frac{dx}{(e^x - x + 1)^2 + \pi^2} = \frac{1}{2}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(e^x - x)^2 + \pi^2} = \frac{1}{1 - W(1)}$$

$W(z)$ is the Lambert function

$$W(z) \ \exp W(z) = z$$

Vardi's evaluation

$$\int_{\pi/4}^{\pi/2} \ln \ln \tan x \, dx = \frac{\pi}{2} \ln \left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \cdot \sqrt{2\pi} \right)$$

uses Dirichlet's L-functions

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots$$

Some integrals of loggamma

L. Euler

$$L_1 := \int_0^1 \ln \Gamma(q)\, dq = \ln \sqrt{2\pi}$$

O. Espinosa and V.M. (2003).

The Ramanujan Journal.

$$\begin{aligned}\int_0^1 \ln^2 \Gamma(q)\, dq &= \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{\gamma}{3} L_1 + \frac{4}{3} L_1^2 \\ &- (\gamma + 2L_1) \frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2}\end{aligned}$$

What about the higher powers?

$$L_3 := \int_0^1 \ln^3 \Gamma(q)\, dq$$

is connected with *Tornheim-Zagier* sums

$$T(a,b;c) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n^a\, m^b\, (n+m)^c}$$

O. Espinosa, V. M. (2006).
Journal of Number Theory.

An interesting new addition

Gradshteyn-Rhyzik 3.248.5:

$$\varphi(x) \;=\; 1 + \frac{4x^2}{3(1+x^2)}$$

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[\varphi(x) + \sqrt{\varphi(x)}\right]^{1/2}} \;=\; \frac{\pi}{2\sqrt{6}}$$

Unfortunately it is wrong

What are these good for?

$$\int_0^1 \frac{(1-x)^4\,x^4}{1+x^2}\,dx \;=\; \frac{22}{7} - \pi$$

so that $\pi \neq \frac{22}{7}$.

Wallis' formula: $\sim$ 1655

$$\frac{1}{\pi}\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = 2^{-2m-1}\binom{2m}{m}$$

What happens for higher degree?

A quartic integral

G. Boros and V. M. (2001).

$$
\begin{aligned}
N_{0,4}(a;m) &:= \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \\
&= \frac{\pi}{2^{m+3/2}\,(a+1)^{m+1/2}} P_m(a)
\end{aligned}
$$

where

$$
P_m(a) = \sum_{l=0}^{m} d_l(m) a^l
$$

with

$$
d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}
$$

The double square root

Theorem: (Boros, G. - V.M., 2001)

$$\sqrt{a+\sqrt{1+x}} = \sqrt{a+1}\times \left(1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\,\frac{P_{k-1}(a)}{2^{k+1}(a+1)^k}x^k\right)$$

Conjecture. There is an *explicit* expression for

$$\sqrt{a+\sqrt{b+\sqrt{1+x}}} = \sum_{n=0}^{\infty}\beta_n[a,b]x^n$$

that involves *homogeneous* versions of $P_m(a)$:

$$T_m(a,b) = b^m P_m\left(\frac{a}{b}\right)$$

Geometric proof of expansion??

The polynomial Riemann hypothesis

The coefficients $d_l(m)$ satisfy

$$d_l(m) = \frac{1}{l!\, m!\, 2^{m+l}} \times$$

$$\left(\alpha_l(m) \prod_{k=1}^{m} (4k-1) - \beta_l(m) \prod_{k=1}^{m} (4k+1) \right)$$

where α_l and β_l are polynomials in m.

Theorem (John Little, 2003). The zeros of the families $\alpha_l(m)$ and $\beta_l(m)$ are on the line $Re(m) = -1/2$.

Arithmetic properties of coefficients

$$
\begin{aligned}
b_l(m) &= \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l} \\
&= 2^{2m} d_l(m)
\end{aligned}
$$

Observe that $b_l(m)$ is even.

What is $\nu_p(b_l(m))$?

Theorem: (G. Boros, J. Shallit, V.M. (2001))

$$\nu_2(b_{0,m}) = s_2(m)$$

(= sum of binary digits of m)

$$\nu_2(b_1(m)) = s_2(m) + \nu_2(m(m+1))$$

Landen transformations

Rational case: degree 6
Boros,G. - V.M. 1997

$$U(a,b;c,d,e) = \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1}\, dx$$

is invariant under the transformation

$$\begin{aligned} a_{n+1} &= T^{-4/3}\,(a_n b_n + 5a_n + 5b_n + 9) \\ b_{n+1} &= T^{-2/3}\,(a_n + b_n + 6) \end{aligned}$$

with $T = a_n + b_n + 2$ and similar formulas for c_n, d_n and e_n.

Higher degree

Theorem. (Boros, G. - V.M., 2000) Math. Comp.

Given an even rational function $R(x)$ there exist an explicit transformation of the coefficients

$$a_{n+1} = F(a_n, b_n, \cdots)$$

that preseves the integral of R.

Convergence

Theorem. (Hubbard, J. - V.M., 2001)
Journal London Math. Soc.

Assume that the initial parameters are such that the integral of R is finite. Then the previous algorithm converges quadratically to the $L/(z^2+1)$.

The original integral can now be evaluated by iteration.

Better proof of convergence

Theorem. (M. Chamberland, J. - V.M., 2006)
Discrete and Continuous Dynamical Systems

The basin of attraction of the fixed point $(3, 3)$ is the (open) region bounded from below by one of the branches of the resolvent curve
$R(a, b) := 4a^3 + 4b^3 - 18ab - a^2b^2 + 27 = 0$

Extensions to the whole line

Theorem. (D. Manna - V.M., 2006)
American Math. Monthly

The integral

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c}$$

is invariant under the transformation

$$a \mapsto \frac{a}{\Delta} \times \left((a+3c)^2 - 3b^2\right)$$

$$b \mapsto \frac{b}{\Delta} \times \left(3(a-c)^2 - b^2\right)$$

$$c \mapsto \frac{c}{\Delta} \times \left((3a+c)^2 - 3b^2\right)$$

where

$$\Delta = (3a+c) \times (a+3c) - b^2$$