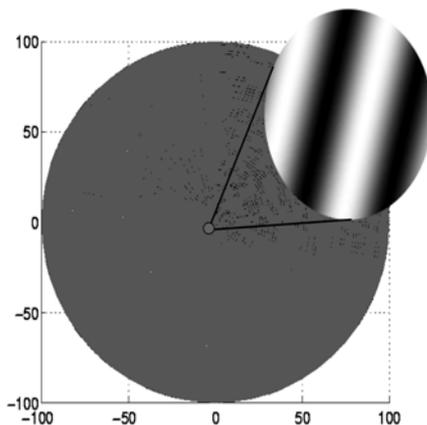


# Exploration and Discovery in Inverse Scattering

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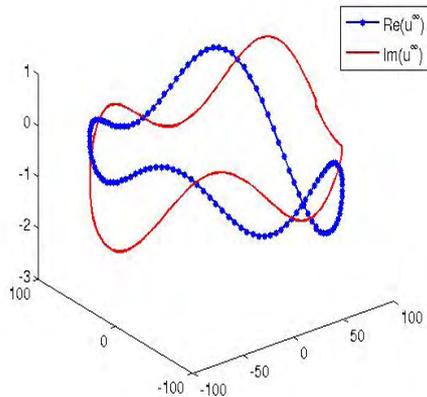
# The Physical Experiment



**Experimental domain:**  $\mathbb{D} \subset \mathbb{R}^2$

**Illuminating source:**  $u^i(x; \hat{\eta}, \kappa) := e^{i\kappa\hat{\eta}\cdot x}$ ,  $x \in \mathbb{R}^2$ , a plane wave where  $\hat{\eta} \in \mathbb{S} := \{d \in \mathbb{R}^2 \mid |d| = 1\}$  is the **incident direction** and  $\kappa > 0$  is the **wavenumber**.

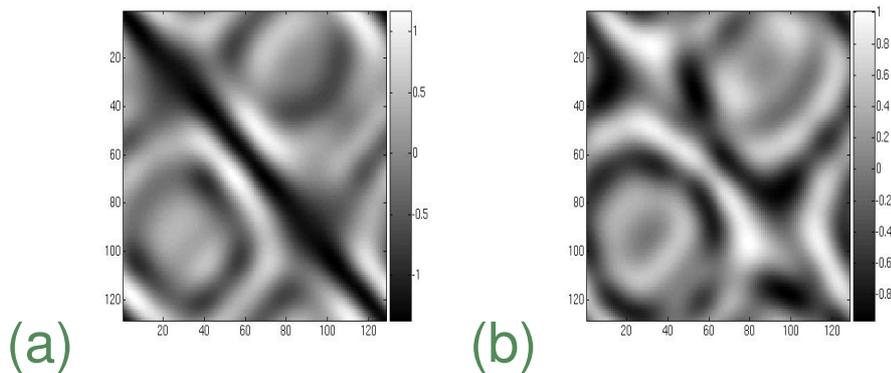
# The Physical Experiment



**Measured data:** far field pattern for the scattered field, denoted by  $u^\infty(\cdot, \hat{\eta}, \kappa) : \partial\mathbb{D} \rightarrow \mathbb{C}$  at points  $\hat{x}$  uniformly distributed around  $\partial\mathbb{D}$ , the boundary of  $\mathbb{D}$ .

## The Physical Experiment

Repeat at  $N$  incident directions  $\hat{\eta}_n$  equally distributed on the interval  $[-\pi, \pi]$ . For each incident direction  $\hat{\eta}_n$ , collect  $N$  **far field measurements** at points  $\hat{x}_n \in \partial\mathbb{D}$ .



Far field data, real (a) and imaginary (b) parts, from 128 experiments differing in the direction of the incident field. Each experiment is at the same incident wavenumber  $\kappa = 2$ .

# The Physical Experiment

**Goal:** determine as much as possible about the scatterer(s) that produced this data, e.g. *where is it?*, *what kind of scatterer is it?* and *what is its shape?*

**Cutting edge research:** *what is the conductivity/permittivity of the scatterer?*

# The Model

Let  $\Omega \subset \mathbb{R}^2$ , be the support of one or more scattering obstacles, each with connected, piecewise  $C^2$  boundaries  $\partial\Omega_j$ .



# The Model

**Model:** Find  $u : \mathbb{R}^2 \rightarrow \mathbb{C}$  that satisfies

$$(*) \quad [\Delta + n(x)\kappa^2]u(x) = 0, \quad x \in \mathbb{R}^2$$

where

$$n(x) := \frac{c_0^2}{c^2(x)} + i\sigma(x), \quad \text{(index of refraction)}$$

with background sound speed  $c_0 > 0$ , scatterer soundspeed  $c : \mathbb{R}^2 \rightarrow \mathbb{R}_+ \setminus \{0\}$ , and absorption  $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ .

This models the propagation of waves in an **inhomogeneous medium**.

## The Model

If the absorption or the scatterer soundspeed take on extreme values of  $\infty$  or 0 respectively, we model the scatterers as impenetrable *obstacles* with one of the boundary conditions

$$\begin{cases} u(x) = 0, & x \in \partial\Omega_j & \text{(a)} \\ \frac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega_j & \text{(b)} \\ \frac{\partial u}{\partial n}(x) + i\kappa\lambda u(x) = 0, & x \in \partial\Omega_j & \text{(c)} \end{cases}$$

on  $x \in \partial\Omega = \bigcap_{j=1}^J \partial\Omega_j$  with unit outward normal  $n$ .

## The Model

The boundary value problem – the Helmholtz equation with (a) Dirichlet, (b) Neumann, and/or (c) Robin boundary conditions – describes a time-harmonic scalar field on the open exterior region  $\Omega^o$  due to the scattering of an incident field  $u^i$  with frequency  $\kappa$  off of (a) sound soft, (b) sound hard, and/or (c) impedance (with impedance  $\lambda$ ) obstacles.

As these are limiting cases, we focus on the case of an **inhomogeneous medium**.

# The Model

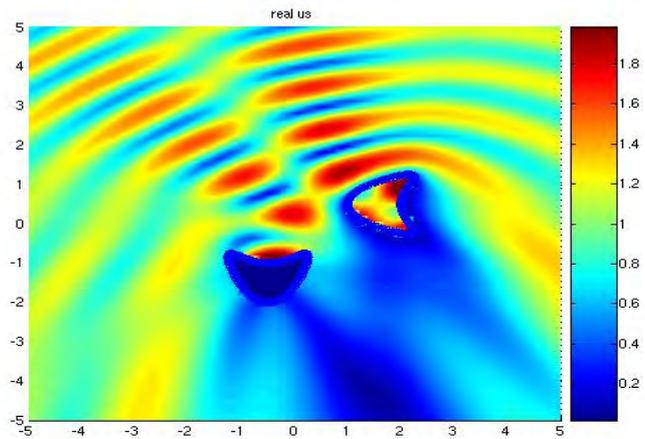
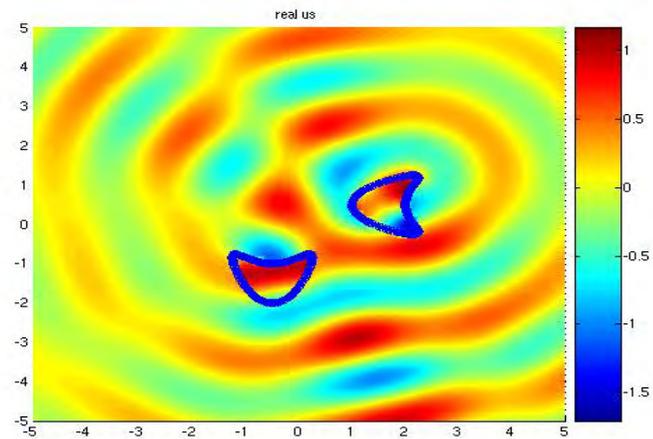
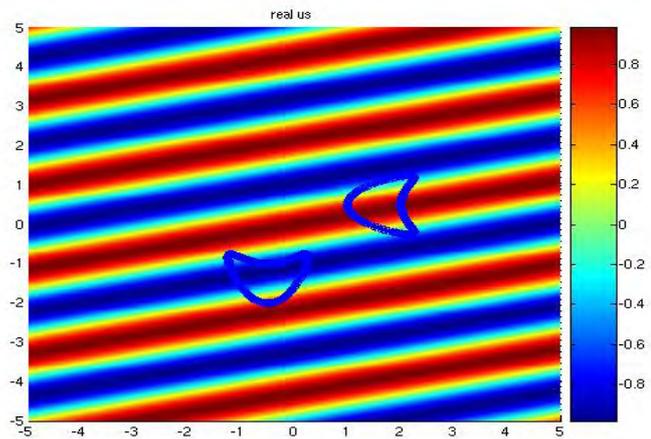
Write  $u = u^i + u^s$  where

the **total field**  $u : \mathbb{R}^2 \rightarrow \mathbb{C}$  solves the Helmholtz equation on  $\mathbb{R}^2$

the **incident field**  $u^i : \mathbb{R}^2 \rightarrow \mathbb{C}$  solves the Helmholtz equation on  $\mathbb{R}^2$

the **scattered field**  $u^s : \mathbb{R}^2 \rightarrow \mathbb{C}$  solves the Helmholtz equation on  $\mathbb{R}^2$ , and  
the **radiation condition**

$$r^{\frac{1}{2}} \left( \frac{\partial}{\partial r} - i\kappa \right) u^s(x) \rightarrow 0, \quad r = |x| \rightarrow \infty,$$



# The Model

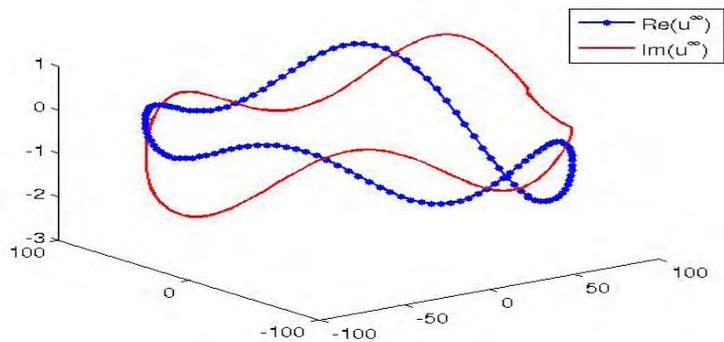
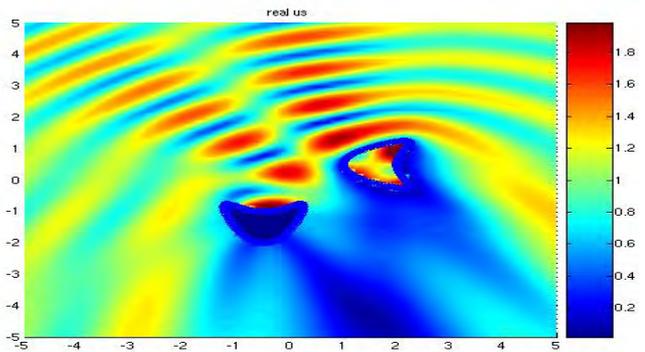
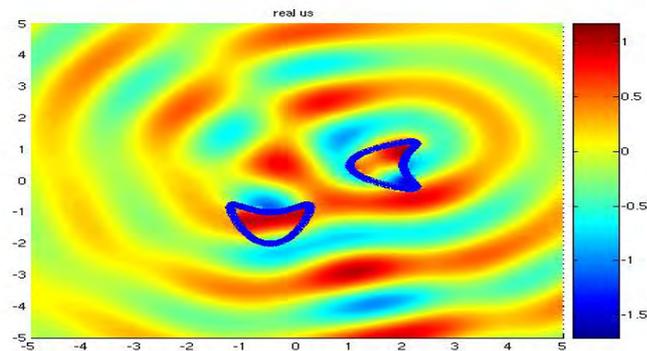
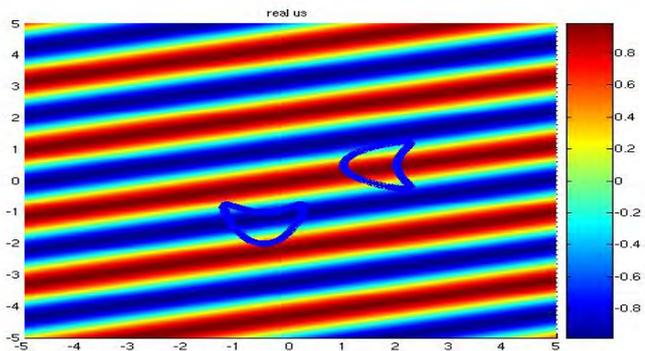
If  $\|x\|$  is very large (the **far field**) then

$$u^s(x, \hat{\eta}) = \beta \frac{e^{i\kappa|x|}}{|x|^{1/2}} u^\infty(\hat{x}, \hat{\eta}) + o\left(\frac{1}{|x|^{1/2}}\right), \quad \hat{x} = \frac{x}{|x|} \quad |x| \rightarrow \infty,$$

where  $u^\infty : \mathbb{S} \rightarrow \mathbb{C}$  is the **far field pattern**,  $\mathbb{S} := \{x \in \mathbb{R}^2 \mid |x| = 1\}$  and  $\hat{x} := \frac{x}{|x|}$ .

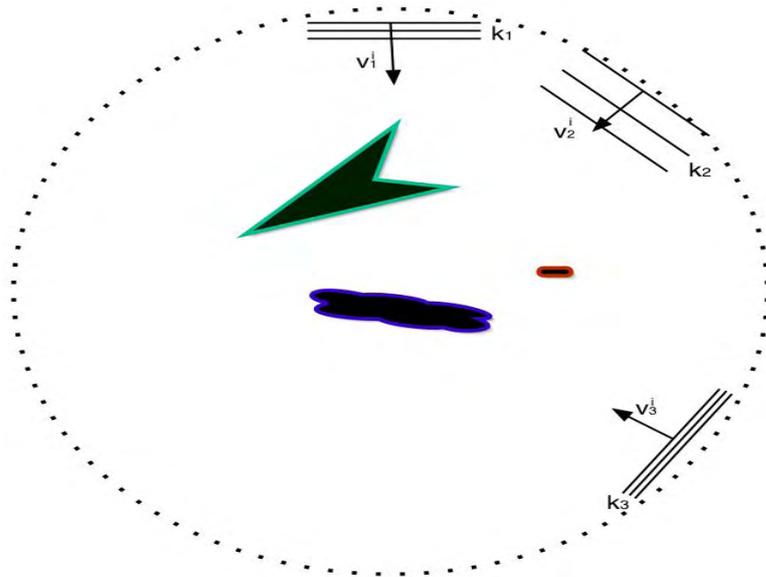
The parameter  $\hat{\eta}$  in the argument of the fields above keeps track of the direction of the incident field.

# The Model



# Problem Statement

Given one or more triplets  $(\kappa, \widehat{\eta}, u^\infty)$ , determine  $\partial\Omega$  and as much information about  $n(x)$ , the **index of refraction**, as possible.



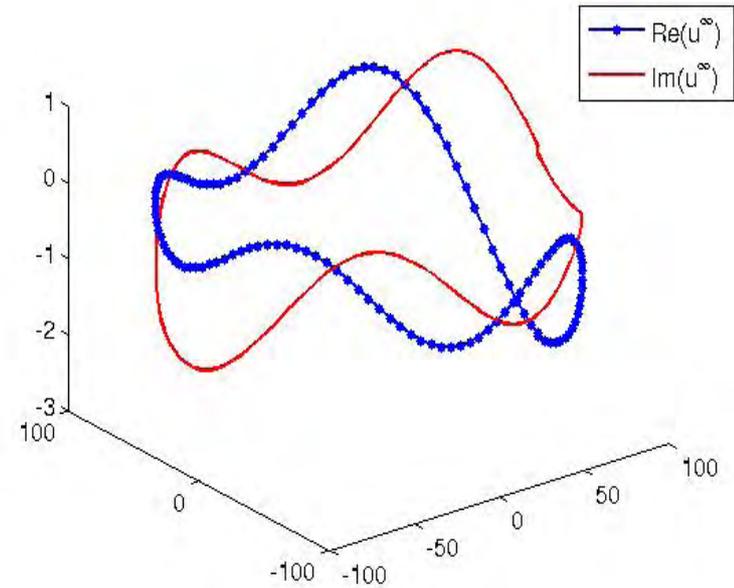
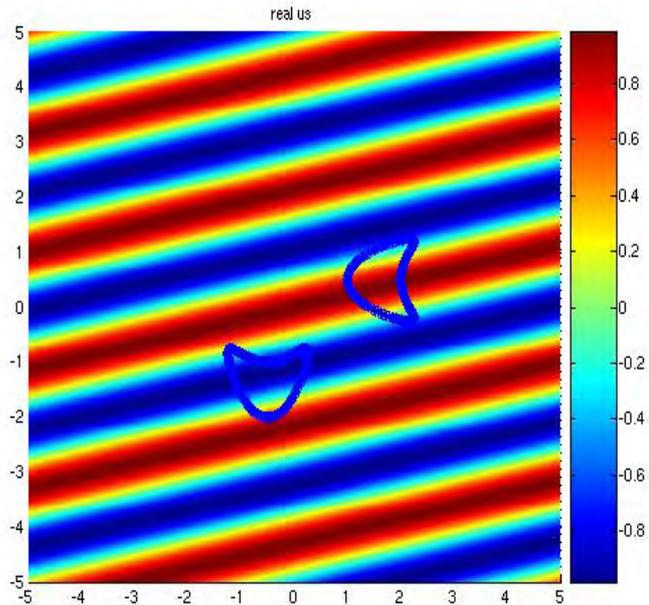
# Problem Statement

For each incident wave  $u^i(\cdot, \hat{\eta})$  at wavenumber  $\kappa$  the solution to the **scattering problem** defines an operator mapping the scattered field on the scatterer  $\Omega$  to the far field pattern  $u^\infty$ :

$$\mathcal{F} : \partial\Omega \rightarrow u^\infty$$

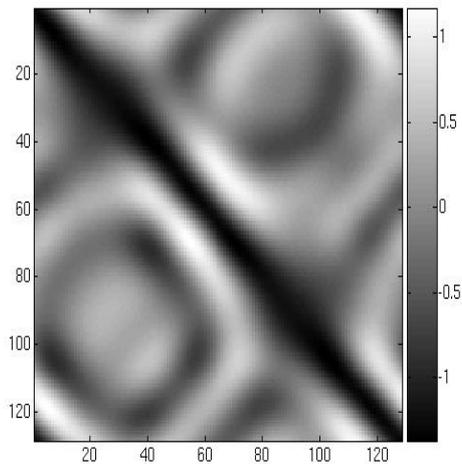
# Problem Statement

For a single incident plane wave, the data would consist of

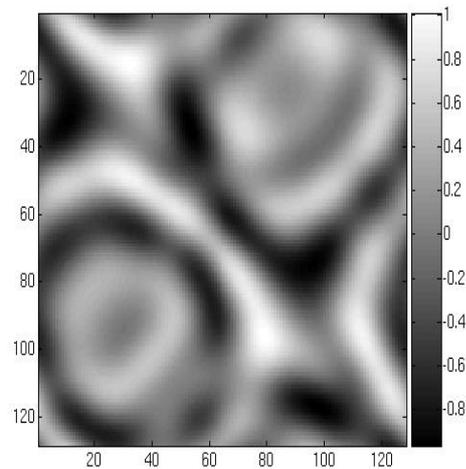


# Problem Statement

For  $N$  incident plane waves, the data is a 2-D array of numbers  $u_{ij}^\infty$  indexing the measurement point  $\hat{x}_i$  in the **far field**, and the **incident field direction**  $\hat{\eta}_j$ .



(a)



(b)

## Where Is the Scatterer? How Big is It?

Define the Herglotz Wave function

$$v_g^i(x, \kappa) := \int_{\mathbb{S}} e^{-i\kappa x \cdot \hat{y}} g(-\hat{y}) ds(\hat{y})$$

(superposition of plane waves). Denote the corresponding scattered and far fields by  $v_g^s$  and  $v_g^\infty$ .

## Where Is the Scatterer? How Big is It?

By linearity and boundedness of the scattering operator, we have

$$v_g^s(x, \kappa) := \int_{\mathbb{S}} u^s(x, -\hat{y}, \kappa) g(-\hat{y}) ds(\hat{y})$$

and

$$\begin{aligned} v_g^\infty(\hat{x}, \kappa) &:= \int_{\mathbb{S}} u^\infty(\hat{x}, -\hat{y}, \kappa) g(-\hat{y}) ds(\hat{y}) \\ &= \int_{\mathbb{S}} u^\infty(\hat{y}, -\hat{x}, \kappa) g(-\hat{y}) ds(\hat{y}) \quad (\text{reciprocity}) \end{aligned}$$

## Where Is the Scatterer? How Big is It?

Given the far field pattern  $u^\infty(\hat{\eta}, -\hat{x})$  for  $\hat{\eta} \in \Lambda \subset \mathbb{S}$  due to an incident plane wave  $u^i(\cdot, -\hat{x})$  with fixed direction  $-\hat{x}$ , let  $v_g^i$  denote the **incident Herglotz wave field** defined above and  $v_g^\infty(\hat{x})$  the corresponding far field pattern. The **scattering test response** for the test domain  $\Omega_t$  is defined by

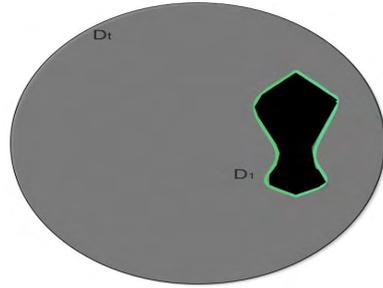
$$\mu(\Omega_t, u^\infty(\Lambda, -\hat{x})) := \sup \left\{ |v_g^\infty(\hat{x})| \mid g \in L^2(-\Lambda) \text{ with } \|v_g^i\|_{\Omega_t} = 1 \right\}.$$

# Where Is the Scatterer? How Big is It?

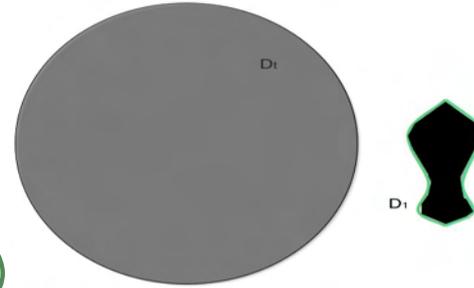
## Theorem [L.&Potthast, 2003]

- (a) If  $\Omega \subset \Omega_t$ , then  $\mu(\Omega_t, u^\infty(\Lambda, -\hat{x})) < \infty$ .
- (b) If, on the other hand,  $\Omega_t \cap \Omega = \emptyset$ , and  $\mathbb{R}^n \setminus \overline{\Omega_t} \cup \overline{\Omega}$  is connected, then  $\mu(\Omega_t, u^\infty(\Lambda, -\hat{x})) = \infty$ .
- (c) If  $\mathbb{R}^M \setminus (\Omega_t \cap \Omega)^c \neq \Omega^c$ , and if the scattered field  $u$  cannot be analytically continued throughout  $(\Omega_t \cap \Omega)^c$ , then  $\mu(\Omega_t, u^\infty(\Lambda, -\hat{x})) = \infty$  [Potthast, 2005].

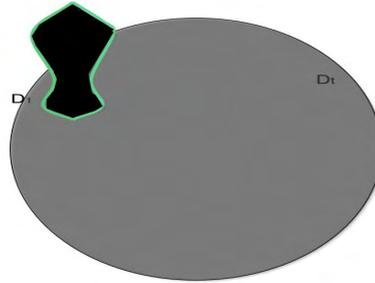
# Where Is the Scatterer? How Big is It?



(a)



(b)



(c)

## Where Is the Scatterer? How Big is It?

Let  $\Omega_0$  denote a fixed, bounded smooth test domain. Denote translations of  $\Omega_0$  by  $\Omega_0(z) := \Omega_0 + z$  for  $z \in \mathbb{R}^2$ . Define the **corona** of the scatterer  $\Omega$ , relative to the scattering test response  $\mu$  by

$$\mathbb{M}_\mu := \bigcup_{z \in \mathbb{R}^2} \Omega_t(z). \\ \text{s.t. } \mu(\Omega_t(z), u^\infty(\Lambda, -\hat{x})) < \infty$$

**Theorem [ Approximate size and location of scatterers L.& Potthast, 2003]**

The scatterer  $\Omega$  is a subset of its corona,  $\mathbb{M}_\mu$

# Where Is the Scatterer? How Big is It?

## Advantages:

- Requires only one incident field and
- the boundary condition is irrelevant

## Disadvantages:

- involves solving an infinite dimensional optimization problem at each translation point  $z$

## Where Is the Scatterer? How Big is It?

Let  $\Omega_0(0)$  be a circle of radius  $r$  centered at the origin where  $r$  is large enough that  $\Omega \subset \Omega_0(z)$  for some  $z \in \mathbb{R}^2$ . For each  $\hat{y} \in \mathbb{S}$ , let  $g(\hat{\eta}, r\hat{y}, 0)$  solve

$$\int_{\mathbb{S}} e^{-i\kappa x \cdot \hat{\eta}} g(-\hat{\eta}, r\hat{y}, 0) ds(\hat{\eta}) = \Phi(x, r\hat{y}), \quad x \in \partial\Omega_t(0).$$

Define the *partial scattering test response*,  $\delta : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ , by

$$\delta(z) := \int_{\mathbb{S}} \left| \int_{\Lambda} e^{-i\kappa \hat{\eta} \cdot z} u^\infty(\hat{\eta}, -\hat{x}) g(-\hat{\eta}, r\hat{y}, 0) d\hat{\eta} \right| ds(\hat{y}), \quad \Lambda \subset \mathbb{S}$$

## Where Is the Scatterer? How Big is It?

**Conjecture.** For any  $\hat{x} \in \mathbb{S}$  there exist constants  $0 < M' < M$  such that

$$\delta(z) \begin{cases} > M & \forall z \in \mathbb{R}^2 & \text{where } \Omega \cap \Omega_t(0) + z = \emptyset \\ < M' & \forall z \in \mathbb{R}^2 & \text{where } \Omega \subset \text{int}(\Omega_t(0) + z). \end{cases}$$

## Where Is the Scatterer? How Big is It?

**Details.** The equation

$$\int_{\mathbb{S}} e^{-i\kappa x \cdot \hat{\eta}} g(-\hat{\eta}, r\hat{y}, 0) ds(\hat{\eta}) = \Phi(x, r\hat{y}), \quad x \in \partial\Omega_t(0).$$

is ill-posed, albeit linear, with respect to  $g$ .

Write the integral operator as  $\mathcal{H}$ .

We regularize the problem by solving the regularized least squares problem

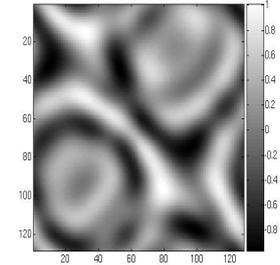
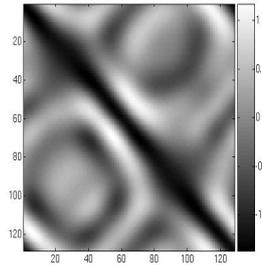
$$\underset{g \in L^2(\mathbb{S})}{\text{minimize}} \quad \|\mathcal{H}g - \Phi^\infty(\cdot, r\hat{y})\|^2 + \alpha \|g\|^2.$$

which yields

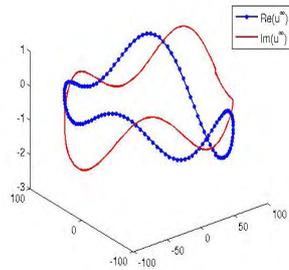
$$g(\cdot; r\hat{y}, 0) \approx (\alpha I + \mathcal{H}^* \mathcal{H})^{-1} \mathcal{H}^* \Phi^\infty(\cdot, r\hat{y}).$$

# Where Is the Scatterer? How Big is It?

For our data set, we take one column of

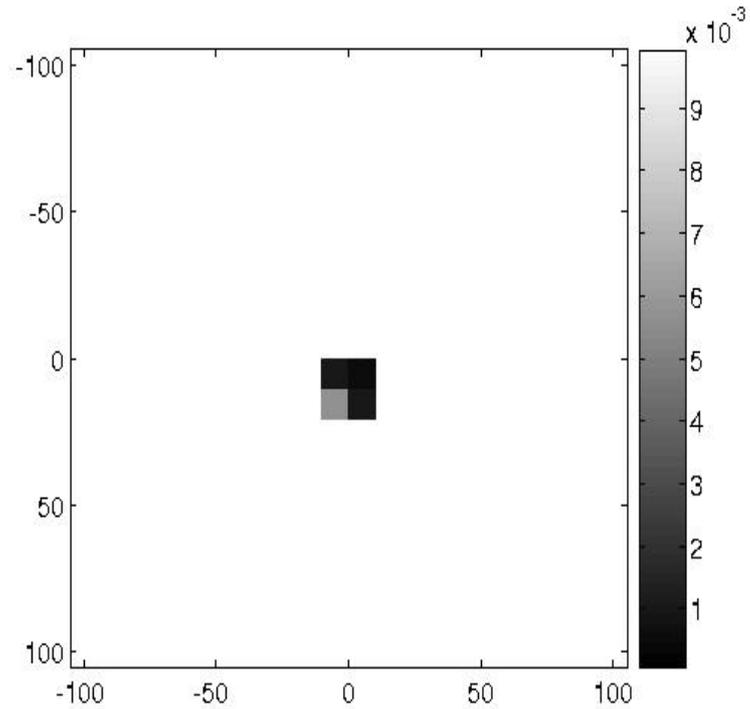


corresponding to



and perform the above test to yield

# Where Is the Scatterer? How Big is It?



## Is the Scatterer Absorbing?

To answer this question, we study the spectral properties of the **far field operator**:

$$\mathcal{A}f(\hat{x}) := \int_{\mathbb{S}} u^\infty(\hat{x}, -\hat{\eta}) f(-\hat{\eta}) ds(\hat{\eta}).$$

**Fact:** There exists some  $g(\cdot; z) \in L^2(\mathbb{S})$  such that  $u^s(z, \hat{\eta}) = (\mathcal{A}g(\cdot, z))(\hat{\eta})$ , i.e. the scattered field lies in the range of the far field operator.

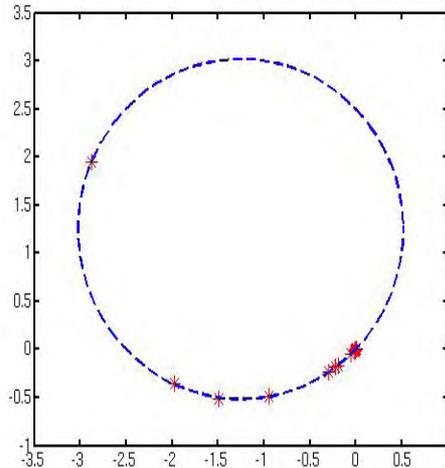
$\implies$  the spectrum of the far field operator should say **something** about the types of scattered fields that can be generated, and hence something about the nature of the scatterer(s).

## Is the Scatterer Absorbing?

**Theorem**[Colton&Kress]. Let the scattering inhomogeneity have index of refraction  $n$  mapping  $\mathbb{R}^2$  to the upper half of the complex plane. The scattering inhomogeneity is nonabsorbing, that is,  $\text{Im}n(x) = 0$  for all  $x$ , if and only if the eigenvalues of  $\mathcal{A}$  lie on the circle centered at  $\frac{1}{2\kappa} (\text{Im}(\beta^{-1}), \text{Re}(\beta^{-1}))$  and passing through the origin. Otherwise, the eigenvalues of  $\mathcal{A}$  lie on the interior of this disk.

Requires all incident directions on  $\mathbb{S}$  and all far field measurements on  $\mathbb{S}$ .

## Is the Scatterer Absorbing?



The eigenvalues (asterisks) of the far field matrix shown line up on the circle passing through the origin with center  $1/2\kappa(\text{Im}\beta, \text{Re}\beta)$  for  $\kappa = 2$  and  $\beta$  a known constant. This implies that the inhomogeneity is nonabsorbing.

## What Is the Shape of the Scatterer?

We will construct an indicator function to determine the feasibility of an auxiliary problem that allows us to tell whether a point is inside or outside the scatterer.

## What Is the Shape of the Scatterer?

$$\begin{aligned} (*) \quad \Delta w(x) + \kappa^2 n(x)w(x) &= 0, & \Delta v(x) + \kappa^2 v(x) &= 0 \text{ for } x \in \text{int}(\Omega) \\ (**) \quad w - v &= f(\cdot, z), & \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} &= \frac{\partial f}{\partial \nu} \text{ on } \partial\Omega. \end{aligned}$$

**(1a) Fact [uniqueness].** If the medium is absorbing, that is  $\text{Im}(n(x)) > 0$ , then there are no nontrivial solutions to the homogeneous problem (\*)-(\*\*) with  $f = 0$ , hence the inhomogeneous problem will have a unique solution **when a solution exists**.

## What Is the Shape of the Scatterer?

From the previous experiment, however, our medium is nonabsorbing, i.e.  $\text{Im}(n(x)) = 0$  for all  $x$ , so there is still the threat of nonuniqueness.

**(1b) Fact [uniqueness, Colton and Päivärinta (2000)].** The set of values of  $\kappa$  for which the solution to (\*)-(\*\*) with  $f = 0$  has a nontrivial solution – called **transmission eigenvalues** – is a discrete set. Hence, for almost all  $\kappa$  (\*)-(\*\*) has a unique solution, **if it exists**.

## What Is the Shape of the Scatterer?

**(2) Theorem [existence, Kirsch].** Let

$$f(y) = h_p^{(1)}(\kappa|y|)Y_p(\hat{y}), \quad (1)$$

a spherical wave function of order  $p$ . The integral equation

$$\int_{\mathbb{S}} u^\infty(\hat{x}; \hat{y}) g(-\hat{x}) ds(\hat{x}) = \frac{i^{p-1}}{\beta\kappa} Y_p(\hat{y}), \quad \hat{y} \in \mathbb{S} \quad (2)$$

has a solution  $g \in L^2(\mathbb{S})$  if and only if there exists  $w \in C^2(\text{int}(\Omega)) \cap C^1(\Omega)$  and

a function  $v$  given by

$$v(x) = \int_{\mathbb{S}} e^{i\kappa x \cdot (-\hat{y})} g(-\hat{y}) ds(\hat{y}) \quad (3)$$

such that the pair  $(w, v)$  is a solution to (\*)-(\*\*)

## What Is the Shape of the Scatterer?

**(3) Theorem [Colton&Kress]** There exists a unique weak solution to (\*)-(\*\*) with  $f(x; z) := \Phi(x, z)$  for every  $z \in \text{int}(\Omega)$  with  $\Phi$  the free-space fundamental solution, that is, the pair  $(w, v)$  satisfies

$$w + \kappa^2 \int_{\mathbb{R}^M} \Phi(x, y)(1 - n(y))w(y) dy = v \quad \text{on } \text{int}(\Omega) \quad (4)$$

and

$$- \kappa^2 \int_{\mathbb{R}^M} \Phi(x, y)(1 - n(y))w(y) dy = \Phi(x, z) \quad \text{for } x \in \partial\mathbb{B} \quad (5)$$

where  $\mathbb{B} \subset \mathbb{R}^2$  is a ball with  $\text{int}(\Omega) \subset \mathbb{B}$ .

## What Is the Shape of the Scatterer?

**Recap.** Equation (2) has a solution if and only if there is a corresponding solution to (\*)-(\*\*) with  $f$  given by (1); moreover, (\*)-(\*\*) with  $f = \Phi(x, z)$  is solvable for every  $z \in \text{int}(\Omega)$ .

**Question:** for  $z \in \mathbb{R}^2 \setminus \Omega$ , or, just as  $z \rightarrow \partial\Omega$  from  $\text{int}(\Omega)$ , what happens to solutions to

$$\int_{\mathbb{S}} u^\infty(\hat{x}; \hat{y}) g(-\hat{x}) ds(\hat{x}) = \Phi^\infty(\hat{y}, z), \quad \hat{y} \in \mathbb{S} \quad (6)$$

where  $\Phi^\infty(\hat{y}, z)$  is the far field pattern of the fundamental solution  $\Phi(y, z)$ ?

## What Is the Shape of the Scatterer?

**Linear Sampling.** For every  $\epsilon > 0$  and  $z \in \Omega$  **there exists** a  $g(\cdot; z) \in L^2(\mathbb{S})$  satisfying

$$\left\| \int_{\mathbb{S}} u^\infty(\hat{x}, -\hat{\eta}) g(-\hat{\eta}) ds(\hat{\eta}) - \Phi^\infty(\cdot, z) \right\|_{L^2(\mathbb{S})} \leq \epsilon \quad (7)$$

such that

$$\lim_{z \rightarrow \partial\Omega} \|g\|_{L^2(\mathbb{S})} = \infty. \quad (8)$$

# What Is the Shape of the Scatterer?

**Details.** The equation

$$\int_{\mathbb{S}} u^\infty(\hat{x}, -\hat{\eta}) g(-\hat{\eta}) ds(\hat{\eta}) = \Phi^\infty(\cdot, z)$$

is ill-posed, albeit linear, with respect to  $g$ .

Write the integral operator as  $\mathcal{A}$ .

We regularize the problem by solving the regularized least squares problem

$$\underset{g \in L^2(\mathbb{S})}{\text{minimize}} \|\mathcal{A}g - \Phi^\infty(\cdot, z)\|^2 + \alpha \|g\|^2.$$

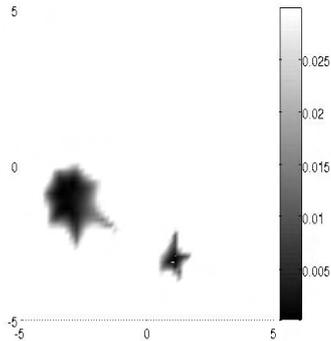
which yields

$$g(\cdot; z, \alpha) := (\alpha I + \mathcal{A}^* \mathcal{A})^{-1} \mathcal{A}^* \Phi^\infty(\cdot, z).$$

## What Is the Shape of the Scatterer?

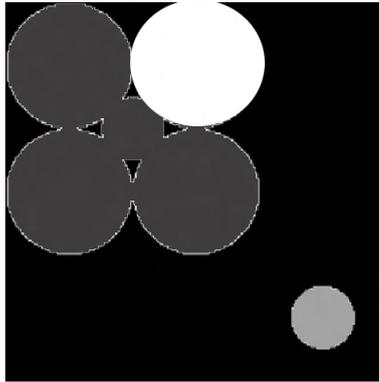
Since we already know from the scattering test response approximately where and how big the scatterer is, we needn't calculate  $g(\cdot; z, \alpha)$  at all points  $z \in \mathbb{D}$ , but rather just on the corona  $\mathbb{M}_\mu$ , or, if we are confident of the earlier Conjecture, then on the corona of the partial scattering response  $\mathbb{M}_\delta$  for a given  $\delta$ . We identify the boundary of the scatterer by those points  $z_j$  on a grid where the norm of the density  $g(\cdot; z_j, \alpha)$  becomes large relative to the norm of the density at neighboring points.

## What Is the Shape of the Scatterer?



Shown is  $\|g(\cdot; z_j, \alpha)\|_{L^2(\mathbb{S})}$  for  $g(\cdot; z_j, \alpha)$  the regularized density with  $\alpha = 10^{-8}$  for all grid points  $z_j$  on the domain  $[-6, 6] \times [-6, 6]$  sampled at a rate of 40 points in each direction. The cutoff is 2.

## What Is the Answer?



The true scatterer consisting of 6 circles of different sizes and indices of refraction indicated by the color.

## Current Research

The methods we have reviewed are not exhaustive. There are numerous variations and alternative strategies. Current research is tending toward meta-algorithms that use different techniques in concert in order to progressively tease more information out of the data. Questions about **what constitutes an image**, and **object discrimination** are fundamental to the science of imaging. In many cases specific knowledge about a very special case can allow for easy discrimination, however this likely does not generalize. The methods we illustrated here are applications of general principles that can be applied in a wide variety of settings. Research on extending these principles to other physical applications is ongoing. What will not change about these methods is the constant interplay between computational experimentation and analysis.