Does Occam's razor shave too close ?  
Or is fate such a great provider ?

« I have myself always thought of a mathematician as in the first instance an observer, a man who gazes at a distant range of mountains and notes down his observations. »

« Beauty is the first test. There is no permanent place in the world for ugly mathematics. »

*Godfrey H. Hardy* (1877-1947)

« I have the result, but I do not yet know how to get it. »

*Carl F. Gauss* (1777-1855)
Does Occam's razor shave too close? Or is fate such a great provider?

A reminder of

**observations presented at the ICM 1998**

(ad-hoc short com):

1) The fixed point of the function

\[ \text{Ahmes}(x) := \left( \frac{x+1}{x} \right)^{(x+1)} \]

equals \( \pi \) with a 0.02\% error, and the point at infinity is \( e \).

2) The fixed point of

\[ \text{Rhind}(x) := \left( 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^{(x+1)}} \right)^4 \]

approximates \( \pi \) to 0.0001\%!

3) 3 + 1/8 on 6 bits (i.e. 11.0010), completed by the **first 19 bits of e written from right to left** equals \( \pi \) with a 0.000001\% error (25 bits in common)!

Why are these numbers so close to \( \pi \)? And why is there still no explanation 8 years later?
The aim of this presentation

1) To present remarkable new observations.

2) To convince you that it is worth trying to explain some of them with mathematical arguments even when random chance seems enough

Research must not be discouraged by the heuristic maxim called:

**Occam's razor principle**
*(the principle of parsimony)*

which says:

*Do not introduce unnecessary assumptions or hypotheses in the explanation of a phenomenon.*

**Keep It Short and Simple**

English logician and Franciscan friar William of Ockham (1285-1349?)
Brother William (The Name of the Rose, Umberto Eco) a Franciscan monk in 1327, inspired by the character of William of Occam.
OUTLINE

1) It is worth looking for mathematical explanations.

2) Coincidences in the first bits of $\pi$ and $e$.

3) The occurrence of remarkable strings of digits, like '01234567' in the decimals of $1/81$, can be explained.

4) An elementary unexplained observation: the Champernowne constant $0.123\ldots101112131415\ldots$

5) A dozen of unexplained occurrences starting from:

\[
\frac{\pi + 2}{\pi + 1} \approx \frac{\log(\pi + 1)}{\log\pi} = 1.24141\ldots \\
\text{and} \quad \sqrt{2} = 1.41421\ldots
\]

6) The simplest conclusion is not always the right one: A very long string of zeros after the decimal point.
The first 19 bits of e in the sequence of the binary digits of π

\[ e = \ldots 010.10110111111000010 \ldots \]

\[ \pi = \ldots 000 \ 11.0010 \ 0100011111101010101 \ 000 \ldots \]

The first occurrence of these 20 bits written from right to left is at position 6 in π!

The second is at position 1,948,205 (≈ almost 2 million).

The length of the circle of radius 1 is:

\[ 2 \pi = 6 + 1/2^2 + \varepsilon_1 \]

and \( \varepsilon_1 \approx 1/30 \)

begins with the first 19 bits of e in reverse.
20 bits of $e$ in the first bits of $2\pi - (5/2)^2$ and $2\pi / (5/2)^2$

The period of the exponential function is $2\pi i$

\[ e^{2\pi i} = 1 \]

\[ 2\pi = 25/4 + \varepsilon_1 = (25/4) (1 + \varepsilon_2) \]

$\varepsilon_1$ has 19+1 bits in common with $e$ (in reverse).

$\varepsilon_2$ has 13 bits in common with $e$ followed by the first 13 bits of $2\pi$ (or $\pi$):

\[ 2\pi = 110.010 \ 01000011111101101010 \ldots \]

\[ 25/4 \quad \leftarrow \quad \approx e \ (20 \text{ bits}) \]

\[ 1 + \varepsilon_2 = 1.0000000101011011111111100100100001 \ldots \]

\[ \approx e \ (7 \text{ zeroes } + 13 \text{ bits}) \quad \approx 2\pi \ (13 \text{ bits}) \]

\[ e = \ldots 000000 \ 010.10110111111000010 \ldots \]

\[ \rightarrow \]
Looking for a series to explain the 19 bits of e (in reverse) at the 6th position of the binary development of π:

**40 bits of π in the first digits of a number close to e**

\[ e_{\pi^{40b}} = e - e/2^{20} + e^2/2^{30} - 1/2^{31} = \]

\[ 010.1011011111000010010011000000000000000010000..._2 \]

14 zeros followed by the first 26 bits of π

This number in base 10 (recreational mathematics):

\[ e_{\pi^{40b}} = 2.718279242519414142... \]

\[ \sqrt{2} = 1.4142... \]

(“1414” at position 14) (5 digits in common)

\[ C_b = 0.123...31415161718...2425...40.4142... \]

\[ \sqrt{2} - 1 = 0.4142 \]

(The Champernowne constant)
Some coincidences do have mathematical explanations

In base b=10, the number \( \frac{b}{(b-1)^2} = \frac{1}{8.1} = 0.12345679 \ 012345679 \ 0... = 0 + \frac{1}{(8 + \frac{1}{(9 + \frac{1}{1}})) = [0; 8, 9, 1] \)

has the same 7 first digits that the normal and transcendental Champernowne constant:

\[ C_{10} = 0.123456789 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15..._{10} \approx C_{10, \text{cf17}} \]

\[ C_{10, \text{cf17}} = [0; 8, 9, 1, 149083, 1, 1, 4, 1, 1, 3, 4, 1, 1, 1, 15] \]

These 7 digits match because the decimals of \( \frac{b}{(b-1)^2} \)

begin with the first \( b-3 \) digits of the base of numeration.

In the same way, in base \( b \), the number \( \frac{b^2}{(b^2-1)^2} \) is written:

\[ 00.01 \ 02 \ 03... \ 0(b-2) \ 0(b-1) \ 10 \ 11 \ 12 \ 13 \ 14 \ 15... (b^2-3) \ (b^2-1) \ 00 \ 01 \ 02..._b \]

It has \( 2(b^2-b-1) \) digits in common with the Champernowne number at position \( b-2 \).

So, in base \( b=10, \ 1 / 98.01 \) has 178 digits in common with \( C_{10} \) at position 8 (see appendix).
Some coincidences do not (yet ?) have a mathematical explanation

In every base \( b > 8 \), the **Champernowne constant** begins in the same way:

\[
C_b = 0.12345678 \ldots 10111213 1415161718 \ldots\]  
= \[b \text{ digits of base } b\]  
= \[b \text{ digits of base } b\]  
= \[(b+1) \text{ digits of base } b\]  
= \[(b+1)^2 \text{ digits of base } b\]

\[
\approx \pi_{10} = \phi_{10} = (e-1)_{10} \approx \sqrt{2}_{10}
\]

1.415\ldots = [1; 2, 2, 2, ...]
1.617\ldots = [1; 1, 1, 1, 1, 1, 1, ...]  
The golden ratio in base 10 is \( \phi_{10} = 1.6180 \ldots \)
1.718\ldots = [1; 1, 2, 1, 1, 4, 1, 1, ...]

Recreational:
In base 10, "3 1415 161 718" occurs between 2 remarkable positions:

<table>
<thead>
<tr>
<th>position</th>
<th>(16 = 2^4 = 2^{2^2} = \lfloor 16.18... \rfloor = \lfloor 10 \phi \rfloor)</th>
</tr>
</thead>
<tbody>
<tr>
<td>and position</td>
<td>(27 = 3^3 = \lfloor 27.18... \rfloor = \lfloor 10 e \rfloor)</td>
</tr>
</tbody>
</table>

The notation \( \lfloor \ldots \rfloor \) means the integer value.
6 independent remarkable approximated formulas waiting for an explanation

\[ 0.12414... = \frac{\pi + 2}{\pi + 1} \approx \frac{\log(\pi + 1)}{\log \pi} = 0.124141... \]

5 digits in common with log(\pi + 1) = 1.4210...

and with \( \sqrt{2} \) or \( \sqrt{2} - 1 \)

1 / (\pi + 1) = 0.2414...

\( \sqrt{2} + 1 = 2.4142... \)

\( \pi + 2 = 5.1415... \)

\( \left( 4.1410414 / 3.14104141421 \right)^{4.14104141421} = 3.1410414... \approx \pi \)
A 7th and 8th coincidence related to
\[(\pi + 2) / (\pi + 1) \approx (\log(\pi + 1)) / \log\pi\]

Ah(x) := \((x+1)/x\)^{(x+1)} \rightarrow e \quad \text{and} \quad \text{Ah}^\infty \approx \pi

Ah^\infty = 3.141 041 525 410 ... is the solution of the equations:

Ah(x) = x \quad \text{(the fixed point of the Ahmes function)} \quad \text{i.e.}

\((x+2) / (x+1) = (\log(x+1)) / \log x\) \quad \text{i.e.}
\((x+2) / \log(x+1) = (x+1) / \log x\) \quad \text{but}

\((x+2) / \log(x+1) \sim (x+1) / \log x \sim \pi(x) \quad \text{(the prime number theorem)}\)

The deviation at \(x = \text{Ah}^\infty\) is:

\[(\text{Ah}^\infty + 2) / \log (\text{Ah}^\infty + 1) - \pi(\text{Ah}^\infty) = 1.61803 ... \approx \varphi = 1.61803 3... \quad \text{(6 digits in common !)}\]
A 9th and 10th coincidence related to

\[
\frac{\pi + 2}{\pi + 1} \approx \frac{\log(\pi + 1)}{\log \pi}
\]

\[
\log (\pi + 1) = 1.4210 \ldots \text{ has 5 digits in common (in reverse) with}
\]

\[
\frac{\log(\pi + 1)}{\log \pi} = 0.124141 \ldots \quad \text{but}
\]

\[
\log(e + 1) = 1.313 \ldots \text{ has all its digits in common with}
\]

\[
\frac{\log(e + 1)}{\log e}
\]

\[
\frac{e + 2}{e + 1} = 1.2689412136999 \ldots \text{ has 5 digits in common with}
\]

\[
\frac{\log(\pi + 1)}{\log \pi} \quad \text{and} \quad (\pi + 2) / (\pi + 1) = 0.12414\ldots \quad \text{(the bold ones)}
\]

\[
\text{and 6 (almost 7) with } \sqrt{2} = 1.41421356 \ldots \approx 1.4142136
\]
3 more coincidences related to

\[ \frac{\pi + 2}{\pi + 1} \approx \frac{\log(\pi + 1)}{\log \pi} \]

\(\frac{\pi + 2}{\pi + 1} = \underline{1.2414} \ldots\)

has 5 digits in common (in reverse) with

\(\sqrt{2} = \underline{1.41421} \ldots\)

\((\text{and } \frac{\log(\pi + 1)}{\log \pi} = \underline{1.2414} 1 \ldots)\) but

\(\approx \sqrt{2} \ (6 \text{ digits})\)

\(\frac{\sqrt{2} + 2}{\sqrt{2} + 1} = \sqrt{2}\)

has all its digits in common with \(\sqrt{2}\).

\(\frac{e + 2}{e + 1} = 1.2689 \underline{41421} 36 \ldots\)

has 6 (and almost 7) digits in common with

\(\sqrt{2} = \underline{1.41421} 356 \ldots \approx \underline{1.41421} 36\)

\(\frac{1/\phi + 2}{1/\phi + 1} = \frac{1}{\phi} + 1 = \phi = \underline{1.6180339} \ldots\)

\(\frac{A_{\infty} + 2}{A_{\infty} + 1} = \underline{1.2414} 851418088 \ldots = \frac{\log(A_{\infty} + 1)}{\log(A_{\infty})}\)

\(\approx \sqrt{2} - 1 \leftarrow (4 \text{ and almost 5 digits of } \pi = 3.14159 \ldots)\)

(The first 3 and almost 4 digits of \(\sqrt{2} = 1.41\ldots\))
The simplest hypothesis is not always the right one

We defined \( C_{10, cf17} = 120.999 \, 998 \, 998 / 980.1 \approx 1 / 8.1 \)

Let \( v_0 = 1 + C_{10, cf17} - C_{10} \)

and \( v_1 = 8.1 \sum_{k > 0} \lfloor k \cdot v_0 \rfloor / 10^k \)

So, \( v_1 = 1.0000000000...000... \) and the computer prints as many zeros as it can!

Is this number an integer? No!

But it has more consecutive zeros after the decimal point than there are Planck cubes (with edges of Planck length \( \approx 1.616 \times 10^{-35} \) m) in the observable universe (of radius: 13.7 billions of light years \( \approx 13.7 \times 10^9 \times 3 \times 10^8 \) m/s \( \times 3.15...\times10^7 \) s), i.e. more than \( 10^{183} \).

And do not miss the observation of the first non zero decimal because it is followed by another pack of more than \( 10^{183} \) zeros.

(A proton is approximately \( 10^{-15} \) m >> Planck length)
A quickly decreasing sequence towards 1

$$v_1 \approx 1$$  Almost true identities: "High-precision frauds"
as J. Borwein and D. Bailey call them.

$$v_{n+1} = 8.1 \sum_{k > 0} \left\lfloor k \cdot v_n \right\rfloor / 10^k$$

The second element of this sequence,
$$v_2 = 1.0000000000000...$$  has more than $$10^{10^{183}}$$
consecutive zeros after the first "1".

The third element ...

We constructed a decreasing (lower bounded) sequence
that converges towards 1 extremely quickly.
(similar ideas are found in ref. [3]).
CONCLUSION

The most exciting phrase to hear in science, the one that heralds new discoveries, is not « Eureka! » (I found it!) but « That's funny... »

Isaac Asimov
(1920-1992)
REFERENCES


WEB SITES (1)

http://www.expmath.info : Experimental Mathematics (with a list of websites)

http://pari.math.u-bordeaux.fr : PARI/GP software

http://www.cecm.sfu.ca/projects/ISC : Centre for Experimental and Constructive Mathematics

WEB SITES (2)

http://pi.nersc.gov or pisearch.lbl.gov:
Search the 4 billion bits of Pi for a string

http://www.research.att.com/~njas/sequences:
The On-Line Encyclopedia of Integer Sequences

http://mathworld.wolfram.com/AlmostInteger.html:
Eric Weisstein’s World of Mathematics
CREDITS

CREATIS (CNRS UMR 5515, INSERM U 630, INSA, Lyon), Centre de Recherche et d'Applications en Traitement de l'Image et du Signal, Research and Applications Center in Image and Signal Processing, Isabelle MAGNIN, Director, and Pierre SEILLE who used the technical specifications of the author of these slides to write the code of the DigITallOccurrences software (to count and locate strings of characters or digits in large texts or numbers in any base of numeration).

Laboratoire A2X (CNRS UMR 9936, Université Bordeaux I) for the powerful PARI/GP free calculator software (GNU General Public License), Karim BELABAS, Henri COHEN, etc...

SNECMA (SAFRAN Group, Paris), Information Systems Department.

SMF (Société Mathématique de France),
AMS (American Mathematical Society),
Marie VANOLI.

THANKS TO
If you have

**explanations** for these observations, **remarks** or **questions** about this presentation, **information**, or **analogous observations** to communicate,

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APPENDIX

Occurences and
Additional observations
A better precision
Previous presentations
Occurrences of 19 bits of $e$ in $\pi$ and reciprocally

The first occurrence of the digital reflection of the first 19 bits of $e$ in the sequence of the binary digits of $\pi$ is at position 6.

$\pi = \ldots 000\ 11.00\ 10\ 010000111110110101 01000\ldots$

$\ldots 01110\ 110\ 100\ 010000111110110101 1\ldots$

The second is at position 746 913,

the third at position 1 948 205 and the fourth at position 2 388 593.

The first occurrences of the first 19 bits of $e$ are at position 446 009, 758 111, 860 841 and 1 117 446 of the binary development of $\pi$.

The first occurrence of the first 19 bits of $\pi$ is at position 568 313 in the bits of $e$, and in reverse, at position 685 719.
Other occurrences of "314"  
(recreational)

The first non trivial zero of the Riemann zeta function $\zeta(s) := \sum_{k > 0} \frac{1}{k^s}$ is $\frac{1}{2} + 14.13... i$.

The position of the zeroes of $\zeta$ is used in the demonstration of the prime number theorem (see slide 12).

\[ \approx -i^2 / 1.414...^2 + 14.14... i \]

\[ = -1 / (\sqrt{-2})^2 + 10 \sqrt{-2} \]

\[ \pi = 3.141592653589... \]

\[ e^\pi \approx 2 \, 3.14 \, 0693 \approx 2 \, 3.141 \approx \log 2 \approx \pi \]

\[ e^\pi = \pi + 10 + 9.999099979189... \approx 10 \]

\[ \log 2 = 0.69 \, 314 \, 718 \, 0... \approx \pi \approx e^{-2} = 0.718 \, 28 \, 18 \, 28 \, 4... \approx \pi \approx 1 / \phi \]
The simplest hypothesis is not always the right one: the first digits of $\pi$ with exceptions

A truncated Gregory's series for $\pi$:

$$4 \sum_{0 < 2k < 10000001} (-1)^{k-1} / (2k-1) = 3.141592453589793238464643383279502 \ldots$$

These 32 underlined digits are similar to that of $\pi$ at the same positions. But do not infer that the two not underlined "4" are digits of $\pi$ either: The digit at position 7 in the decimal development of $\pi$ is a "6", the digit at position $21 = 3 \times 7$ is a "2".

See more in ref. [1] and [4].
Precisions (1)

In base $b=10$, the rational number $b^2/(b^2-1)^2 = 100/99^2 = 1/98.01 = 0.010203...08091011...97990010203...$

has 178 digits in common with the Champernowne constant at position 8:

$C_{10} = 0.123...891011...979899100101102...$

and all its digits except the first ones are those of the rational number

$C_{10, cf17} = 0.123...891011...97990010203...08091011...$

$$= 604.999999499/4900.5 = (1 - 10^{-8} \times 1.002/1.21)/8.1$$

$$= 10/9^2 - 10^{-6} \times 10.02/99^2$$

which has its 185 first digits in common with $C_{10}$:

$C_{10, cf17} - C_{10} = 0.9101019... \times 10^{01-0190}$

$\leftrightarrow \quad \rightarrow \quad$ (recreational mathematics)

$e_{\pi40b} = 2.718279242519414142875...$

$2.414213... = [2; 2, 2, 2, 2, 2, 2, 2, ...] = \sqrt{2} + 1$
Precisions (2)

1.4151... = \([1; 2, 2, 2, 4, 5, ...]\)
1.6171... = \([1; 1, 1, 1, 1, 1, 2, 1, 2, ...]\)
1.7181... = \([1; 1, 2, 1, 1, 4, 1, 1, 1, ...]\)  
(extracts of the Champernowne number in base 10)

\[
10 \div (\pi + 1) = 2.41453... = \ [2; 2, 2, 2, 2, 1, 5, 117, ...] \approx \sqrt{2} + 1
\]

\[
(\pi + 2) \div (\pi + 1) = 01.2414 \ 53...
\]

\[
\log(\pi + 1) = 1.4210 \ 80412...
\]

\[
\frac{1.41421...}{1.14210...} = \sqrt{2}
\]

\[
\frac{\log(\pi + 1)}{\log\pi} = 01.24141 \ 11227120 \ 0800286 \ 48897 \ 70515 \ 67002 \ 12014 \ 58888 \ 36628 \ 21338 \ 227120 \ 43...
\]

\[
\log\pi = 1421.0 \ ... \ / \ 01241. \ ... = 1.14...
\]

\[
\rightarrow \quad \leftarrow
\]
Other interesting values of \( \frac{x+2}{x+1} \) and \( \frac{\log(x+1)}{\log(x)} \)

There are many other interesting values, for example when \( x \to +\infty, \quad x \to 0, \quad x \to +1 \),

or \( x = 2, \quad x = \phi \):

\[
\frac{2 + 2}{2 + 1} = \frac{4}{3} = 1.33333... \approx 1.33133... = \pi^{1/4} \quad \text{(Scribe Ahmes, 1650 bc)}
\]

\[
\frac{0 + 2}{0 + 1} = \frac{\log(\phi + 1)}{\log \phi} = 2 = (\sqrt{2})^2
\]

...
Similarities in digit sequences in $\pi$ and $e$

Correspondances entre des séquences de chiffres de $\pi$ et $e$

Coincidences exist between certain of the digit sequences of universal numbers. Are the corresponding approaching likelinesses due to pure chance or do they stem from an intelligible structure?

Examples:
in binary numeration, $\pi$ bits from the 7th to the 25th, read from right to left, correspond to the first 19 bits of $e$.

For the function $\text{Ahmespb50}(x) = \left( \frac{x+1}{x} \right)^{(x+1)}$ whose limit is $e$ when $x$ tends towards infinity, the fixed point is

$3.1410... = \pi / 1.00017...$

The fixed point of

$\text{Rhindpb50}(x) = \left( 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^{(x+1)}} \right)^4$

is

$3.1415898... = \pi / 1.000 000 88...$

A natural parameterization of the graph of this function leads to a point of abscissa

$3.14159294... = 1.000 000 093... \pi$
Franco-Canadian congress of Toulouse in 2004, extract from the author's poster

“Why so close to $\pi$?”

A partir du problème 50 du papyrus Rhind copié par le scribe Ahmès 1650 ans avant J.C., voici une approximation de $\pi$ datant de 1850 ans avant notre ère :

$\text{Ahmès pb} 50(3) = \left(\frac{4}{3}\right)^4 = 3,16...$

$\text{Ahmès pb} 50(\text{infini}) = e$ proche de $2 \sqrt{2}$ et de $\pi$

Le point fixe de la fonction $\text{Ahmès pb} 50(x) = \left(\frac{(x+1)}{x}\right)^{(x+1)}$ est la racine de l'équation $(x+1)^{(x+1)} = x^{(x+2)}$ et vaut

$3,141 0 41 ...$ c'est-à-dire $\pi$ à $1 / 5000$ près :

$-1 + 4,141 0 41 ... = \left(\frac{4,141 041 ...}{3,141 041 ...}\right)^{4,141 041 ...}$

$\sqrt{2} = 0 1,414 ...$ et $n^2 + n + 41$ est premier pour $n = -40$ à $40 - 1$

$\text{Ahmès pb} 50(\pi) = \left(\frac{(\pi +1)}{\pi}\right)^{\pi +1)} = \text{environ} 3,141$

$\left(\frac{(\pi +1)}{\pi}\right)^{(\pi +1)} = 3,1415926 1..$ et $\pi = 3,1415926 5...$

$1 + 1 / (\pi + 1 + 1/(2^9 e)) = 1,24141 112 5...$

$log(\pi + 1) / log \pi = 1,24141 112 2... = log 4,141... / log 3,141...$

et, écrits de droite à gauche,

$...124141$ sont les 6 premiers chiffres de $\sqrt{2}$
Le point fixe de la fonction

\[ \text{Rhindpb50}(x) = \left( 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^{(x+1)}} \right)^4 \]

vaut \( 3,1415898... = \pi / 1,000,000,88... \)

Posons \( a(x) = 4 \arctan \left( \frac{\text{Rhindpb50}(x)}{x} \right) \quad x > 0 \)

Alors, le minimum (avec dérivée nulle) de la fonction

\[ F(x) = (x - a(x)) \left( \text{Rhindpb50}(x) - a(x) \right) \]

vaut \( 3,1415929... = \pi / 1,000,000,09... \)
3+1/8 sur 6 bits, suivis par les 19 premiers bits de e écrits de droite à gauche

\[ e = 010,10110111111000010 \quad 1010001011000... \]
\[ e / 2^{20} = 000,000000000000000000000 \quad 010,1011011... \]

et la différence entre les bits écrits ci-dessus s'écrit avec 33 bits de \( \pi \) de droite à gauche :

\[ 010,10110111111000010 \quad 010 \quad 0110000000 \quad 1 \]

car \( \pi \) s'écrit :\[ 000000011,0 \quad 010 \quad 0100001111110110101010... \]

Autre observation récréative :
Les 19 premiers bits de \( e \) sont positionnés entre les positions 25 et 6 des premiers bits de \( \pi \). Or les premiers bits de \( \pi \) correspondent aux valeurs 6 = 110,0 en base 2, 25 = 11001,00 en base 2, et 19 = 010011 en base 2, où l'on reconnaît les 6 premiers bits de \( \pi \) de droite à gauche.