A Few Mathematical Experiments

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Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company,
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With a little help from my friends

Bill Gosper

Dean Hickerson

Dan Hoey

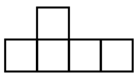
Cheryl Beaver

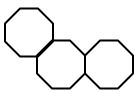
Allan Wechsler

and the Lifers

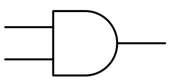
A Few Mathematical Experiments

Li_2 Modular Dilogarithms

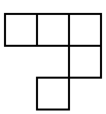
 Counting PolyHypercubes

 Counting PolyPolygons

001101 The Post Tag Problem

 A Boolean Quickie -- Minimal Circuits

3.14159… UltraPrecision & UltracomputableNumbers

 Conway’s Game of Life

Li_2

Modular Dilogarithms

with Bill Gosper & Cheryl Beaver

and kibbitzing from

Don Zagier & Herbert Gangl

Paper at www.cs.arizona.edu/~rcs

Li_2

Classical Dilogarithm

$$Li_2(z) = z + \frac{z^2}{4} + \frac{z^3}{9} + \frac{z^4}{16} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t}\, dt$$

Power series converges when $|z|<=1$

Analytic extension for larger z.

Li_2

Functional Equations

$$Li_2(z) + Li_2(1-z) = pi^2/6 - \ln z \ln(1-z)$$
$$Li_2(z) + Li_2(1/z) = - pi^2/6 - ½ \ln (-z)^2$$

$$Li_2(z) + Li_2(-z) = ½ Li_2(z^2)$$
$$Li_2(z) + Li_2(wz) + Li_2(w^2z) = Li_2(z^3) /3$$

Range reduction $|z| < .8$

Li_2

Spence, Hill, …

$$Li_2(xy) = Li_2(x) + Li_2(y) + Li_2\left(\frac{xy-x}{1-x}\right) + Li_2\left(\frac{xy-y}{1-y}\right) + \tfrac{1}{2}\ln\left(\frac{1-x}{1-y}\right)^2$$

[5term]

[6term]

$$2\,[cLi_2(x) + cLi_2(y) + cLi_2(z)] = cLi_2(xy) + cLi_2(xz) + cLi_2(yz)$$

where $cLi_2(x) = Li_2(1-x)$ and $x+y+z = xyz+2$.

Many more multivariate identities.

Wojtkowiak recently showed 5term implies the others.

Li_2

Modular Dilogarithms:A Shot in the Dark

Is there a modular version of the dilogarithm that satisfies the same algebraic identities?

\+ analogy with discrete logarithms
\- no rational values except 0

What would this mean? How to find it?
Domain: mod P Range: ???
Riemann sheets? Infinity? -- punt

Li_2

Logarithm $\rightarrow$ Discrete Log

$\log_{10} 2 = .30103\ldots$ $\qquad$ $\log_{10} 2 = 17$

mod 19 $\qquad$ mod 18

Dilogarithm $\rightarrow$ Modular Dilog

$Li_2(1/2) = .58224\ldots$ $\qquad$ $D(1/2) = D(10) = 265$

mod 19 $\qquad$ mod 360

Li_2

Results

Mod 19

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
D(N)	120	30	345	358	74	26	344	258	327	108	265	162	3	132	326	44	236	232	345

Mod 360

Solutions mod P for P = 5 … 23.

Answers always mod P^2-1.

Nonprime moduli:

Mod 25 -- answers mod 600 = 24 * 25.

GF[5^2] -- answers mod 624 = 24 * 26. %%

Li_2

Functional Equations

$D(x) + D(1-x) \;=\; D(0) + D(1) + K \log(x) \log(1-x)$

$2\,[\,D(x) + D(1/x)\,] \;=\; -2\,D(1) + K \log(-x)^2 \qquad x/=0$

$2\,[\,D(x) + D(-x)\,] \;=\; D(x^2)$

$D(xy) \;=\; D(x) + D(y) – D((x-xy)/(1-xy)) – D((y-xy)/(1-xy))$
$\qquad + D(0) + K \log((1-x)/(1-xy)) \log((1-y)/(1-xy)) \qquad xy/=1$

$2\,[\,D(1-x) + D(1-y) + D(1-z)\,] \;=\; D(1-xy) + D(1-xz) + D(1-yz)$
with $z = (2-x-y)/(1-xy)$ and $xy/=1$.

K is a multiple of P+1. Logs are discrete logs.
Log 0 is always multiplied by log 1.
Full solution Mod 19: $Z_{360} \times Z_3$.

Li_2

Trilogarithms

$$Li_3(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^3} .$$

Many functional equations.
I got one bivariate fneqn to work.
Solutions are mod P^3-1 or 7(P^3-1). P = 5 … 17

Higher polylogs:
Not clear if multivariate fneqns exist after 6.
Relations of rational arguments exist for all orders.

Li_2

Questions

Is this real?

Some unpublished partial results of Weibel.

P <= 59 for related problem – Keith Dennis

Why does it exist?

Simpler functions +-*/ exp log trig hyp ellfn

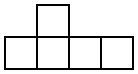
can be reduced to counting.

This floats like Laputa.

How to calculate them?

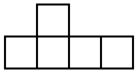
More complicated version of dlog method?

How extensive is it?

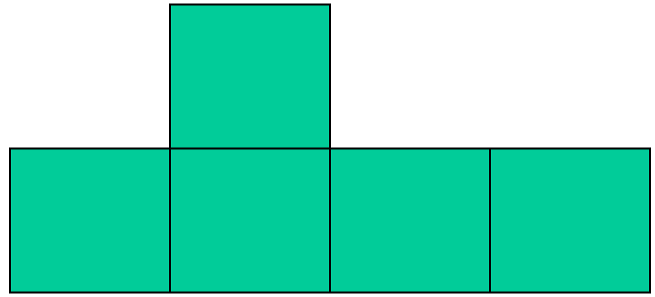


Counting PolyHypercubes

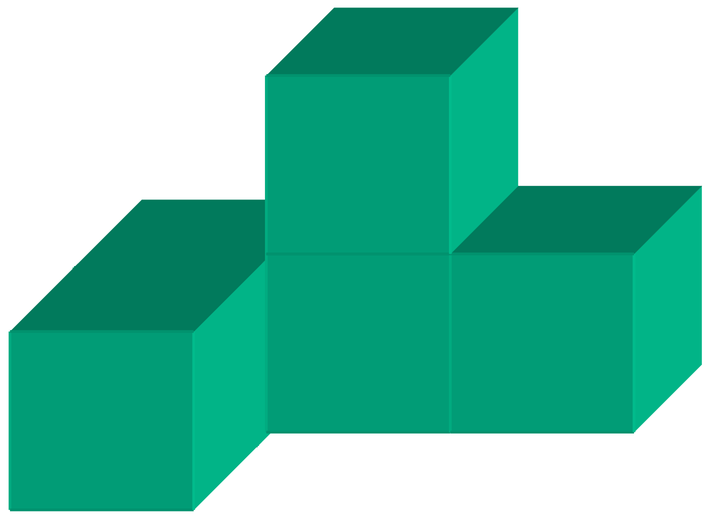
with Dan Hoey & Bill Gosper



Polyomino



Polycube



Polytesseract . . .

So Many Questions, So Little Time

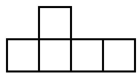
How many polyhypercubes of each dimension & volume?

Asymptotic formula?

Limiting Ratio for adding 1 to the volume?

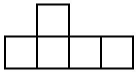
Some way to extrapolate the counts?

How about just getting some data?



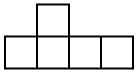
Dimension D, Volume V

including all	multiplies count by roughly
Orientations	$D!$
Reflections	2^{D-1}
Starting Positions	V



PolyHypercubes of Dimension D, Volume V, no symmetries removed

	V	1	2	3	4	5	6	7	8	9	10	11	12	13
D														
-1		1	-2	9	-64	560	-5370	53788	-555864					
0		1	0	0	0	0	0	0	0	0	0	0	0	0
1		1	2	3	4	5	6	7	8	9	10	11	12	13
2		1	4	18	76	315	1296	5320	21800	89190	364460	1487948	6070332	24750570
3		1	6	45	344	2670	20886	164514	1303304	10375830	82947380	665440039		
4		1	8	84	936	10810	127632	1531180	18589840	227826873				
5		1	10	135	1980	30475	483702	7847525	129419240					
6		1	12	198	3604	69405	1386048	28403620	593399416					
7		1	14	273	5936	137340	3307878	81972296	2075032808 poly extrap					
8		1	16	360	9104	246020	6940128	201819688						
9		1	18	459	13236	409185							Dan	Hoey



Columns are Polynomials in D

Volume		PolyHypercube(D)
1		1
2		2 D
3		(12 D^2 – 6 D) / 2
4		(128 D^3 – 180 D^2 + 76 D) / 6
5		(2000 D^4 – 5040 D^3 + 4780 D^2 – 1620 D) / 24
6		(41472 D^5 – 155520 D^4 + 236640 D^3 – 166320 D^2 + 44448 D) / 120
7		1075648 – 5433120 + 11564560 – 12538680 + 6725992 – 1389360
8		33554432 – 214695936 + 592793600 – 880199040 + 723963968 – 305862144 + 50485440

Leading coefficients are $(2DV)^{V-1}/V!$ and $-6\,(2V-3)\,(2V)^{V-4}\,D^{V-2}/(V-3)!$ and $(108V^3-463V^2+1122V-1560)\,(2V)^{V-6}\,D^{V-3}/3\,(V-4)!$ (corrected)

Ratios

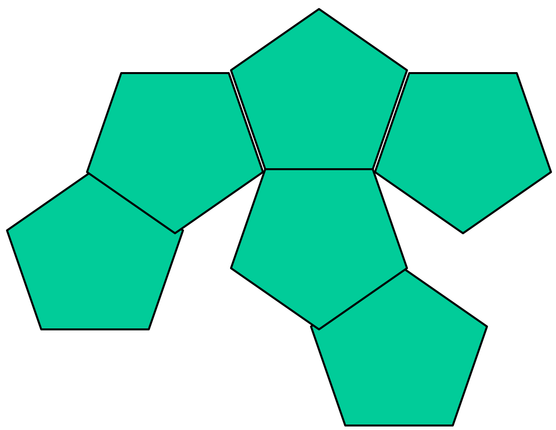
Dimension	Ratio Guess	Tree Bound
		$(2D-1)^{2D-1} / (2D-2)^{2D-2}$
-1	- 10.3	- 9.5 *
0	---	- 4 *
1	1	1
2	4.1	6.75
3	8.0	12.2
4	12.2	17.6
5	16.5	23.1
6	20.9	28.5
7	25.3	34.0
8	29?	39.4
9	31?	44.8
D	4.4 D – 5.5	~ (2D - 1.5) e

PolyHypercubes -- All Symmetries Removed

	V	1	2	3	4	5	6	7	8	9
Dim										
0		1								
1			1	1	1	1	1	1	1	1
2				1	4	11	34	107	368	1284
3					2	11	77	499	3442	24128
4						3	35	412	4888	57122
5							6	104	2009	36585
6								11	319	8869
7									23	951
8										47
Total		1	1	2	7	26	153	1134	11050	128987

PolyPolygons

with Dean Hickerson



No symmetries removed: Position Orientation Reflection

Exact algebraic coordinates of polygons

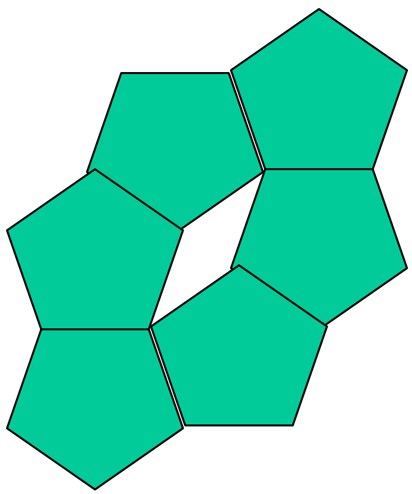
as polynomials in $w = e^{2\pi i/n}$.

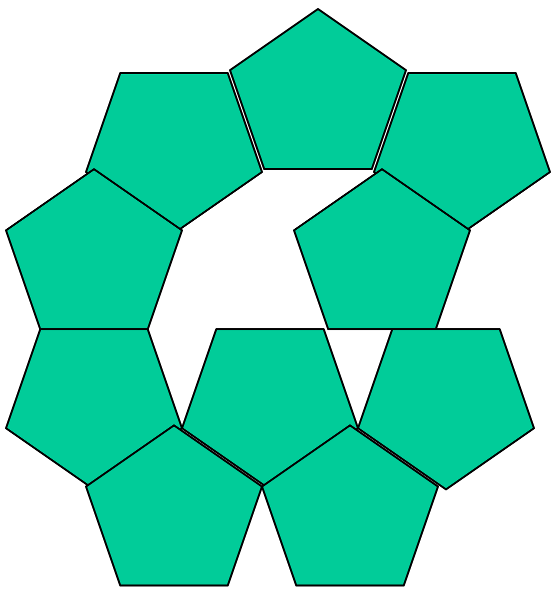
Floating point approximation as first cut for position

determines coarse overlap.

Close cases are resolved by exact computation.

Interesting Special Cases





Edges can meet exactly, but offset sideways.

Bare vertices can touch.

Possible centers are dense in the plane.

PolyPolygons of side S and area A

Area

1	2	3	4	5	6	7	8	9	10
1	3	9	28	90	282	875	2700	8271	25265
1	4	18	76	315	1296	5320	21800	89190	364460
1	5	30	180	1075	6366	37520	220500	1293615	7581240
1	6	33	176	930	4884	25564	133512	696231	3626710
1	7	42	252	1505	8946	53165	315980	1878597	11171930
1	8	60	440	3230	23688	173796	1275240	9359748	68708320
1	9	81	732	6660	60534	549801	4991436	45301356	411062595
1	10	105	1080	11060	112932	1151430	11728960	119405565	1215105280
1	11	132	1584	18920	225258	2677675	31805400	377611443	4481810410
1	12	138	1564	17655	198936	2239860	25209144	283667850	3191677980
1	13	156	1872	22425	268398	3212391	38454000	460400148	5513163565
1	14	189	2520	33530	445788	5925976	78775480	1047251079	13923394730

(side)

PolyPolygons of side S and area A

side \ area	11	12	13	14	15	16
3	77088	235014	716261	2182257	6646200	20234080
4	1487948	6070332	24750570	100868236	410919990	1673486992
5	44398387	259881960	1520633270	8895116230		
6	18876363	98186556	510472118	2652899130		
7	66456082	395399760	2352979538	14004512886		
8	504466468	3704376384	27205146592			
9	3729450978					
10	12361736948					

side \ area	17	18	19	20	21
3	61581327	187366482	569947883	1733389620	5270937735

Ratios

Number of Sides	Estimated Ratio
3	3.04
4	4.07
5	5.84
6	5.19
7	5.96
8	7.34
9	9.07
10	10.17
11	11.86
12	11.25
13	11.98
14	13.29

Checkability

There are a few possible consistency checks. The Area must divide twice each value. In some cases, the number of Sides must divide the value.

The values for polyiamonds, polyominoes, and polyhexes match the Sloane database.

Two different strategies for doing the counting agree, when both exist.

But I really need independent confirmation.

Extrapolation

We need better tools for extrapolating bumpy sequences.

Challenge:

Extrapolate Partition Numbers

$P(n) = 1, 1, 2, 3, 5, 7, 11, \ldots$

What is the Simplest Unsolvable Problem?

3N+1 problem doesn’t count – we know

the answer, even if we can’t prove it.

Here’s a possibility . . .

001101

The Post Tag Problem

with Allan Wechsler

001101

Start with a bit string, and apply this simple rule to it, over and over.

Remove the first three bits, and append 00 or 1101,

depending on the first removed bit.

$$0xx\,\$ \;\; \text{-->} \;\; \$\,00 \qquad = \;\; \$\,A$$

$$1xx\,\$ \;\; \text{-->} \;\; \$\,1101 \qquad = \;\; \$\,B$$

1001

11101

011101

10100

001101

10100

a loop!

What happens to the bit string?

It may grow.

It may shrink.

It will probably do some of both..

It could eventually vanish entirely.

It may eventually repeat itself.

Could it grow to infinity?

Could it simulate a Turing Machine?

More complex Tag systems can.

001101

Looping Strings

0xx --> 00 = A

1xx --> 1101 = B

AB = 00 1101 = 0xx1xx --> AB

BBAA --> BBAA

(AB v BBAA)* --> same

AABBB (AAABBB)* displaced; 1/n – eps passes

001101

Three New Looping Patterns

Period 40	$B^3A^5B^5$	~ 3 passes
Period 66	$AB^2AB^3A^3B^3A^2B^2A^4B^2$	~ 3 passes
Period 282	$AB^3ABABA^2B^2AB^9A^2$	~ 11 passes

Each Looper can have copies of the Full Loop appended

$B^3A^5B^5$ $(B^2ABA^5B^3AB^3A^6B^2ABAB^3A^5B^5)^*$ is also period 40

Some short patterns grow very, very, very long.

Starting Length (Letters)	Maximum Steps *	Longest Intermediate String
1-2	12	6
3-4	28	16
5-6	412	56
7-10	2678	176
11-14	25006	648
15-16	364K	2078
17-22	367M	59468
23-24	2.7G	138056
25-26	>15G	>900000

If there *is* a TM simulation, there should

be some simple patterns that grow linearly.

Maybe, just around the corner?

A Boolean Quickie --
Minimal Circuits for Simple Functions

We want the minimum number of gates to synthesize Boolean functions with a few inputs and one output. To simplify things, we make up some rules:

Only two kinds of logic gates: NOT(x), and Two-Input AND(x,y)

AND gates cost $1 each. NOT gates are free.

Circuits can't have any feedback loops – they must be DAGs.

Fanout is free.

We don't care about delay time through the circuit.

We don't care about area, crossing wires, or power.

Minimal Circuits: Functions of 0 -- 3 Variables

Variables	Function	Gates
0:	0	0
1:	A	0
2:	A&B	1
	A x B	3

Variables	Function	Gates
3:	A&B&C	2
	A&(BxC)	4
	A=B=C	5
	A&(BvC)	2
	(AxB)&(AvC)	4
	A+B+C=1	6
	Ax(B&C)	4
	A(B,C)	3
	Maj(A,B,C)	4
	AxBxC	6

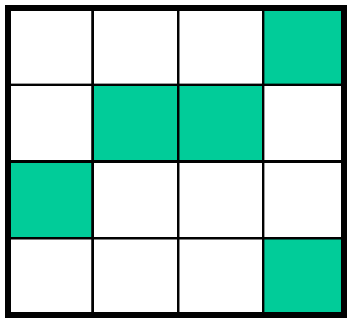
4 Variables

$2^{16} = 65536$ truth tables

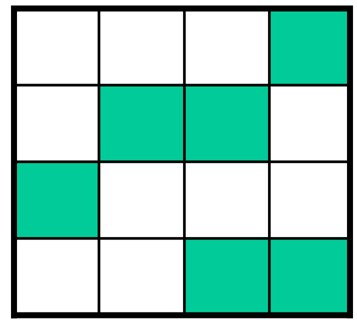
After permuting the inputs, and complementing any of the inputs and perhaps the output, there are 222 distinct functions. 208 use all four inputs.

Number of gates:	3	4	5	6	7	8	9	10
Number of functions:	5	5	23	28	61	45	37	4

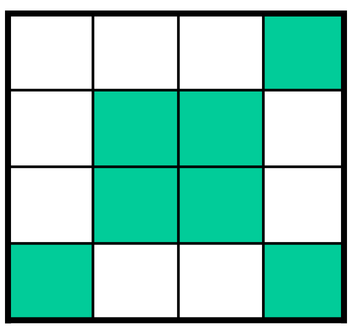
The Four Hardest Boolean Functions of 4 Variables



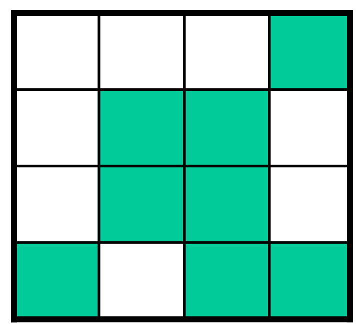
ABCD is 3N+1



ABCD is 0 or 3N+1



A+B+C+D = 0 or 2



Full Adder: A selects bit of B+C+D

Guesses for 5 Variables

2^{32} = 4,294,967,296 truth tables.

616126 different functions.

Maximum gates provably <=23; I’m guessing 16 really.

Computing effort huge – Need better methods.

My search on 4 gates takes 30x for each additional gate.

3.14159…

UltraPrecision

with Bill Gosper & Eugene Salamin

Independent work by Richard Brent, and the Chudnovsky Brothers

999,999,999,999 places of Pi on the Disk,

  999,999,999,999 places of Pi;

Take one down and spin it around,

  999,999,999,998 places of Pi on the Disk.

999,999,999,998 places of Pi on the Disk,

. . .

3.14159…

Ultracomputable Numbers

A number is Ultracomputable if the first N bits beyond the decimal point can be computed in time Tk = O(N logN $^{k+eps}$).
A function is Ultracomputable when its values are Ultracomputable.

+ and - are T0

* and / and sqrt and inverse functions are T1

Sqrt(2) is T1

Pi and e are T2

log and exp and ^ and trig and elliptic fns and radix conversion are T2

gamma, zeta(3), G, $J_0(z)$ and solns of simple Diff Eqns are T3

(simple: sum P(X) $Y^{(n)} = 0$, algebraic coefficients)

Gamma(algebraic) is T3

Walking Riemann surface adds 1 to k, per step.

(Speculative)

Extending the set of Ultracomputable Numbers

To include the values of integrals, and solutions of differential equations.

We approach integrals with an N-point Simpson’s Rule, expecting O(N) bits.

For differential equations, we use a similar method, noting that the various corrections are all linear functions of earlier integration errors.

Computing the Coefficients for a Generalized Simpson's Rule

$$VSC * \text{Vandermonde Matrix} = [1\ \tfrac{1}{2}\ 1/3\ \tfrac{1}{4}\ \ldots\ 1/n]^T$$

The Vandermonde Matrix can be factored into sparse invertible matrices, and moved to the other side of the equation.

The integral we want is VSC * $[F(0)\ F(1)\ \ldots\ F(N)]^T$.

VSC need not be explicitly computed.

Conway’s Game of Life -- Highlights

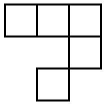
- Stable Objects Counts
- Ratio
- Glider Synthesis
- Spaceship Speeds & Directions
- Synthesis of Spaceships with Gliders
- Oscillator Periods
- Synthesis of Oscillators
- Replicator Work
- Soups: Final densities
- Life is Local in Random Soup
- Low Density Life
- Scores of contributors

Number of Stable Life Objects*

Population	Objects		Population	Objects
4	2		15	1353
5	1		16	3286
6	5		17	7773
7	4		18	19044
8	9		19	45759
9	10		20	112243
10	25		21	273188
11	46		22	672172
12	121		23	1646147
13	240		24	4051711
14	619			

***CORRECTED!**

Ratio = ~ 2.45



Spaceship velocities:

orthogonal: 1/2, 1/3, 1/4, 1/5, 2/5, 1/6, 2/7, 17/45(plan)

diagonal: 1/4, 1/5, 1/12

Oscillator Periods:

everything except 19, 23, 31, 34*, 37, 38, 41, 43, 51*, 53

Random soup final density: ~ 2.8%

weak dependence on starting probability

Locality: Wholesale changes propagate about 500 cells

Life is Local – At Least in the Soup

Experiment:

Two 8K x 8K random starting positions (P = 50%).

Left Halves match; Right Halves completely different.

Run both patterns.

How far does the “difference front” propagate to the Left?

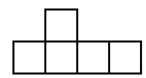
After 20K steps, the furthest DF propagation is 566 cells.

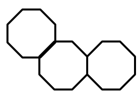
The average DF propagation (over all rows) is 290 cells.

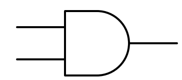
90% of the propagation was in the first 5K steps.

99% was in the first 10K steps.

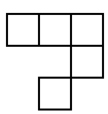
Is ALL of Mathematics Experimental?







Li_2



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